UNIT-I

1. MECHANICS OF PARTICLES

ESSAY QUESTIONS

1. Explain the importance of Newton's laws of motion.

Ans: First law:

Every body continues in its state of rest or of uniform motion in a straight line unless an external force acts on it to change that state.

Significance:

First law states that the body cannot change its state by itself. This property is called inertia. So the first law leads to the definition of inertia. It also states that the force is necessary to change the state the body. Hence it defines the term force as force is one which changes or tries to change the state of the body. The concept of rest, motion or acceleration can be specified only when the frame of reference is specified. Hence Newton's first law is a statement about reference frames. Any frame of reference which obeys the Newton's laws of motion is called inertial frame of reference.

Second law:

The rate of change of momentum of a particle is equal to the force acting on it and takes place in the direction of force.

Significance:

Second law leads to the measurement of force. It states that the time rate of change of momentum of a particle is equal to the force acting on it. Hence

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$$

So the product of mass and acceleration gives the force. When F=0, then a=0. Thus when there is no force the acceleration is zero, so the body will be in uniform motion or at rest. This is the first law. Hence Newton's first law is a special case of second law.

Third law:

To every action there is always an equal and opposite reaction.

Significance:

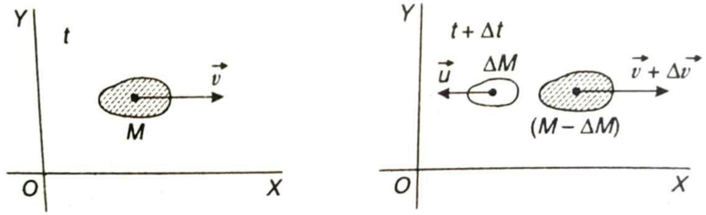
Action and reaction are two forces acting on two different bodies. They never act on the same body. Third law leads to the law of conservation of linear momentum. That is for two bodies, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$.

2. Explain the motion of variable mass system.

Ans: If the mass of the system varies with time, then it is called variable mass system.

Ex: motion of a rocket.

Motion of variable mass system: Consider a system of mass M, whose centre of mass moves with a velocity $\overline{V}$ at any instant of time t. After a time $(t + \delta t)$,a mass δM has been ejected from the system and moves with a velocity $\overline{u}$. The remaining mass $(M - \delta M)$,moves with a velocity $(\overline{V} + \delta\overline{V})$.



Initial momentum of the system = M$\overline{V}$

Final momentum of the system

$$= \delta M\ \overline{u} + (M - \delta M)\left(\overline{V} + \delta\overline{V}\right)$$

Rate of change of momentum

$$= \frac{\left[\delta M\overline{u} + (M - \delta M)\left(\overline{V} + \delta\overline{V}\right)\right] - M\overline{V}}{\delta t}$$

This gives the external force $\overline{F_{ext}}$ acting on the system

$$\therefore \overline{F}_{ext} = \frac{\left[\delta M\overline{u} + (M - \delta M)\left(\overline{V} + \delta\overline{V}\right)\right] - M\overline{V}}{\delta t}$$

$$= \frac{\left[\delta M\overline{u} + M\overline{V} + M\delta\overline{V} - \delta M\overline{V} - \delta M\delta\overline{V}\right] - M\overline{V}}{\delta t}$$

$$= \frac{\delta M}{\delta t}\overline{u} + M\frac{\delta\overline{V}}{\delta t} - \frac{\delta M}{\delta t}\overline{V} - \frac{\delta M\,\delta\overline{V}}{\delta t}$$

When δt approaches to zero,

$$\frac{\delta\overline{V}}{\delta t} \approx \frac{d\overline{V}}{dt},\quad \frac{\delta M}{\delta t} = -\frac{dM}{dt},\quad \frac{\delta M\,\delta\overline{V}}{\delta t} \approx 0$$

$$\therefore \overline{F}_{ext} = -\frac{dM}{dt}\overline{u} + M\frac{d\overline{V}}{dt} + \frac{dM}{dt}\overline{V}$$

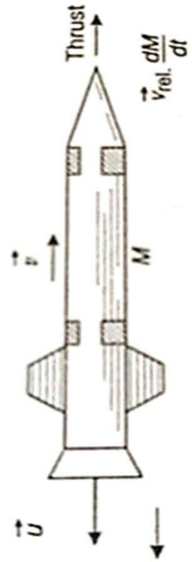
$$\Rightarrow M\frac{d\overline{V}}{dt} = \overline{F}_{ext} + \left(\overline{u} - \overline{V}\right)\frac{dM}{dt}$$

The term $\left(\overline{u} - \overline{V}\right)\frac{dM}{dt}$ is the rate of change of momentum of the system due to the mass leaving it. This can be taken as the reaction force on the system excerted by the mass leaving it.

$$\therefore M\frac{d\overline{V}}{dt} = \overline{F}_{ext} + \overline{F}\ \text{reaction}$$

3. *Explain the motion of a rocket and derive the expression for its velocity.*

Ans: The motion of rocket is an example of system of variable mass. The rocket consists of a combustion chamber in which liquid or solid fuel is burnt. When the fuel is burnt, the pressure inside the combustion chamber rises very high. Due to this high pressure hot gasses are expelled from the tail of the rocket. Then according to the law of conservation of momentum, the rocket moves in the direction opposite to the direction of hot gas jet. As the mass of the fuel inside the rocket decreases with time, the velocity of the rocket increases.



Expression for velocity: Let M be the mass of the rocket including fuel and $\overline{V}$ be its velocity at the time t. Let in a time interval dt, an amount of mass dM be ejected from the rocket in the form of gas jet. Let $\overline{u}$ be the velocity of gas jet relative to the rocket.

Relative velocity = $\overline{V}-\overline{u}$

Force acting on the jet = (rate of change of mass of the rocket) (relative velocity) = $\frac{dM}{dt}\left(\overline{V}-\overline{u}\right)$

From Newton's third law, this is equal to the thrust on the rocket.

$\therefore$ Thrust on the rocket = $\frac{dM}{dt}\left(\overline{V}-\overline{u}\right)$

Weight of the rocket = M $\overline{g}$

Net force acting on the rocket in forward direction

$$= \frac{dM}{dt}\left(\overline{V}-\overline{u}\right)-M\overline{g} \qquad ..... (1)$$

According to Newton's second law,

The net force on the rocket = $\frac{d}{dt}\left(M\overline{V}\right)$ ..... (2)

from (1) & (2) : $\frac{d}{dt}\left(M\overline{V}\right)=\frac{dM}{dt}\left(\overline{V}-\overline{u}\right)-M\overline{g}$

$$\Rightarrow M\frac{d\overline{V}}{dt}+\overline{V}\frac{dM}{dt}=\overline{V}\frac{dM}{dt}-\overline{u}\frac{dM}{dt}-M\overline{g}$$

$$\Rightarrow M\frac{d\overline{V}}{dt}=-\overline{u}\frac{dM}{dt}-M\overline{g}$$

Considering the magnitudes, $M\frac{dV}{dt}=-u\frac{dM}{dt}-Mg$

$$\Rightarrow M\,dV=-u\,dM-Mgdt$$

$$\Rightarrow dV=-u\frac{1}{M}dM-g\,dt$$

Integrating the above equation from the beginning of the motion where t = 0, M = M_0, V = V_0, up to the instant of time where t = t, M = M, V = V.

$$\Rightarrow \int_{V_0}^{V} dV=-u\int_{M_0}^{M}\frac{1}{M}dM-g\int_{0}^{t}dt$$

$$\Rightarrow \left[V\right]_{V_0}^{V}=-u\left[\log M\right]_{M_0}^{M}-g\left[t\right]_0^t$$

$$\Rightarrow V-V_0=-u\log\frac{M}{M_0}-gt$$

$$\Rightarrow V=V_0+u\log\frac{M_0}{M}-gt$$

This gives the velocity of the rocket at any instant of time t.
When the rocket starts from rest, V_0=0,

$$\therefore V=u\log\frac{M_0}{M}-gt$$

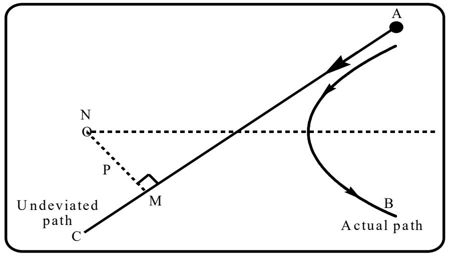
Ignoring the gravity effect, $V=u\log\frac{M_0}{M}$.

4. Explain the working of a multistage rocket.

Ans: The working of the rocket depends on the law of conservation of momentum. According to the law of conservation of momentum, the gas jet emerging in the backward direction makes the rocket to move in the forward direction. The velocity attained by the rocket is nearly 4 km/sec.. To obtain higher velocities, multistage rockets containing two or more stages are used the first stage of the rocket is used to acquire the acceleration of the rocket. When the fuel of the first stage is exhausted, it detaches from rocket and drops off. The velocity at this stage becomes the initial velocity of the second stage. Now the second stage starts functioning. The rocket gains acceleration and its velocity goes on increasing. The removal of the excess mass contained in the first stage considerably helps in attaining the higher velocity. When the fuel of the second stage is exhausted, it also detaches from the rocket. The velocity acquired so far by rocket is less than the escape velocity (11.2 km/sec.). Finally the third stage of the rocket starts with the required velocity.

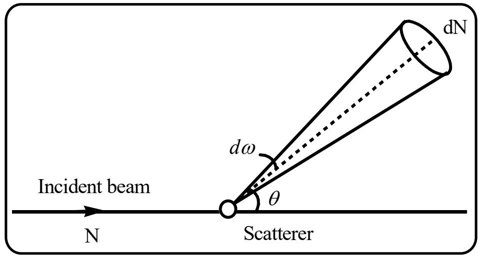
5. Explain the terms impact parameter, scattering cross-section.

Ans: Impact Parameter: Consider a positive particle, like a proton or an α – particle, approaching a massive nucleus N of an atom, as shown in fig. Due to Coulomb force of repulsion, the particles follow a hyperbolic path AB with nucleus N as its focus. In the absence of the repulsive force, the particle would have followed the straight-line path AC. As shown in figure, p is the perpendicular distance from the nucleus N to the original direction AC of the particle. The distance NM = p is called the impact parameter.



Thus impact parameter is defined as the closest distance between nucleus and positively charged particel projected towards it. This is also known as collision parameter.

Scattering Cross-Section: When α – particles are projected into a thin metal foil, they are deflected or scattered in different direc-tions. Let N be

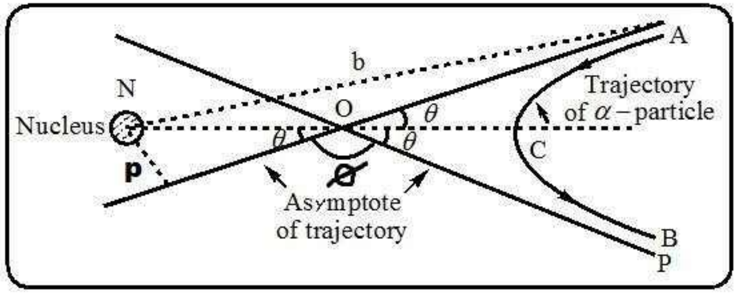


the incident intensity (number of incident particles crossing per unit time a unit surface placed perpendicular to the direction of propagation). Suppose dN be the number of particles scattered per unit time into solid angle $d\omega$ located in the direction θ and ϕ with respect to the bombarding direction. The ratio $\frac{dN}{N}$ is called scattering cross section.

Thus the scattering cross section in a given direction is defined as the ratio of number of scattered particles into solid angle $d\omega$ per unit time to the incident intensity.

6. Explain Rutherford scattering and derive an expression for angle of deviation.

Ans:



Consider a nucleus of charge Ze (Z is atomic number) stationary at a point N and an α – particle of mass m, change 2e and velocity V_0 approaching along the direction AO as shown in fig. As α – particle moves, it comes closer to the nucleus and experiences a repulsive force. With this result, the velocity of α – particle decreases. In the absence of this repulsion, it would have followed a straight path but due to Coulombic force of repulsion, it follows a hyperbolic path AB with nucleus as its focus. The line AO and PO are the asymptotes of the hyperbola and represent approximately the initial and final directions of $\alpha-$particle. In the part of journey AC, the velocity of $\alpha-$particle goes on decreasing, while in the part CB its velocity goes on increasing till it reaches its original velocity V_0. In the fig., p is the impact parameter. The angle of deviation scattering angle is ϕ .

Considering the case when $\alpha-$particle is directed straight towards the nucleus so that p = 0. In this case, the particle will be stopped at a distance **b** from the nucleus due to repulsive force and retraces its path i.e. $\phi = 180^o$. The distance **b** is known as the distance of closest approach.

The electrostatic potential at a distance **b** due to nucleus

$$= \frac{1}{4\pi \in_o}\frac{Ze}{b}.$$

Hence the potential energy of $\alpha-$ particle at a distance **b** from the nucleus = $\left(\frac{1}{4\pi \in_0}\frac{Ze}{b}\right)2e = \frac{2Ze^2}{4\pi \in_0 b}$.

(where 2e is the charge on $\alpha-$particle.)

When the $\alpha-$particle is momentarily stopped at a distance **b**, its kinetic energy $\frac{1}{2}mv_0^2$ is completely converted into its potential energy

$$i.e., \frac{1}{2}mv_0^2 = \frac{2ze^2}{4\pi \in_0 b} \Rightarrow b = \frac{Ze^2}{\pi \in_0 mv_0^2} \quad \text{..... (1)}$$

Consider the case in which $p \neq 0$. In this case the $\alpha-$particle is deflected through an angle which is less than 180^0 and traverses a hyperbolic path. Let V be the velocity of $\alpha-$ particle at the vertex, C.

Angular momentum of $\alpha-$ particle at A = m v_0 P

($\therefore$ Angular momentum = mass × velocity × distance)

Angular momentum of $\alpha-$ particle at C

= mv × (NC) = $m\ v$ d (where NC = d)

$$\therefore m v_0 P = m v d \quad \Rightarrow \quad v = \frac{v_0 p}{d} \quad \text{..... (2)}$$

Kinetic energy of $\alpha-$ particle at A $= \frac{1}{2}mv_0^2$

Kinetic energy of $\alpha-$ particle at C $= \frac{1}{2}mv^2$

Potential energy of $\alpha-$ particle at C

$$= \frac{1}{4\pi \in_0}\left(\frac{Ze \times 2e}{d}\right) = \frac{Ze^2}{2\pi \in_0 d}$$

According to the conservation of energy,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{Ze^2}{2\pi \in_0 d}$$

$$\Rightarrow v^2 = v_0^2 - \frac{Ze^2}{\pi \in_0 md}$$

$$\Rightarrow v^2 = v_0^2 - \frac{bv_0^2}{d} = v_0^2\left(1 - \frac{b}{d}\right) \quad \text{..... (3) from (1)}$$

Substituting the value of v from eq. (2) in eq. (3), we get

$$\frac{v_0^2 p^2}{d^2} = v_0^2\left(1 - \frac{b}{d}\right) = v_0^2\frac{(d-b)}{d}$$

$$\Rightarrow p^2 = d(d-b) \quad \text{..... (4)}$$

From the properties of hyperbola,

Eccentricity of hyperbola = e = sec θ

Focal length ON = e OC = a sec θ (where OC = a)

From figure ON = p cosec θ

$\therefore$ a sec θ = p cosec θ $\Rightarrow$ a = p cot θ

From figure NC = NO + OC

= a sec θ + a= a (1+sec θ)

$\therefore$ d = p cot θ (1 + sec θ) ($\because$ a = p cot θ)

$$d = p\left[\cot\theta + \frac{1}{\sin\theta}\right] = p\left[\frac{\cot\theta\sin\theta + 1}{\sin\theta}\right] = p\left[\frac{\cos\theta + 1}{\sin\theta}\right]$$

$$= p\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = p\cot\frac{\theta}{2}$$

$$\therefore d = p\cot\frac{\theta}{2} \qquad \text{..... (5)}$$

Substituting this value of ***d*** in eq. (4), we have

$$(4) \Rightarrow p^2 = d(d-b)$$

$$\Rightarrow p^2 = p\cot\frac{\theta}{2}\left(p\cot\frac{\theta}{2} - b\right)$$

$$\Rightarrow p = \cot\frac{\theta}{2}\left(p\cot\frac{\theta}{2} - b\right)$$

$$\Rightarrow p = p\cot^2\frac{\theta}{2} - b\cot\frac{\theta}{2} \Rightarrow b\cot\frac{\theta}{2} = p\cot^2\frac{\theta}{2} - p$$

$$\Rightarrow b = \frac{p\left(\cot^2\frac{\theta}{2} - 1\right)}{\cot\frac{\theta}{2}} = p\left(\cot\frac{\theta}{2} - \tan\frac{\theta}{2}\right)$$

$$= p\left[\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right] = p\left[\frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= 2p\frac{\cos\theta}{\sin\theta} = 2p\cot\theta$$

$$= 2p\cot\left(\frac{\pi - \phi}{2}\right) \qquad \text{From fig., } \left(\theta = \frac{\pi - \phi}{2}\right)$$

$$= 2p\cot\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = 2p\tan\frac{\phi}{2}$$

$$\Rightarrow \tan\frac{\phi}{2} = \frac{b}{2p} \qquad \text{..... (6)}$$

Substituting the value of **b** from eq. (1) in eq. (6) we get

$$\tan\frac{\phi}{2} = \frac{Ze^2}{2\pi \in_0 m v_0^2 p} \qquad \text{..... (7)}$$

This equation gives the scattering angle ϕ.

2. MECHANICS OF RIGID BODIES

7. *Define rigid body. Derive rotational kinematics relations.*

Ans: *A body which does not undergo any change in shape or size by the application of external forces is called rigid body.*

A rigid body can be defined as a solid and fixed. The distance between the particles is not disturbed by any external forces applied.

Rotational Kinematics Relations:

Consider a rigid rotating body with initial angular velocity ω_o. Let ω be its angular velocity after a time ‘t’.

Let θ be the angular displacement and 'α' be its angular acceleration.

I. To derive $\omega = \omega_o + \alpha t$

We have angular acceleration $\alpha = \dfrac{d\omega}{dt} \Rightarrow d\omega = \alpha dt$

Integrating between initial and final values,

$$\int_{\omega_o}^{\omega} d\omega = \int_0^t \alpha dt \qquad \Rightarrow [\omega]_{\omega_o}^{\omega} = \alpha [t]_0^t$$

$$\Rightarrow \omega - \omega_0 = \alpha t \qquad \Rightarrow \omega = \omega_o + \alpha t$$

II. To derive $\theta = \omega_o t + \dfrac{1}{2}\alpha t^2$

Integrating between initial and final values,

$$\int_0^\theta d\theta = \omega_0 \int_0^t dt + \alpha \int_0^t t \text{ dt}$$

$$\Rightarrow [\theta]_0^\theta = \omega_0 [t]_0^t + \alpha \left[\frac{t^2}{2}\right]_0^t$$

$$\Rightarrow \theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

To derive $\omega^2 = \omega_0^2 + 2\alpha\theta$

We have (1) $\Rightarrow \omega = \omega_0 + \alpha t \Rightarrow t = \dfrac{\omega - \omega_0}{\alpha}$

(2) $\Rightarrow \theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$

$$\Rightarrow \theta = \omega_0 \left(\frac{\omega - \omega_0}{\alpha}\right) + \frac{1}{2}\alpha \left(\frac{\omega - \omega_0}{\alpha}\right)^2$$

$$\Rightarrow \theta = \frac{\omega_0\omega - \omega_0^2}{\alpha} + \frac{\omega^2 + \omega_0^2 - 2\omega\omega_0^2}{2\alpha}$$

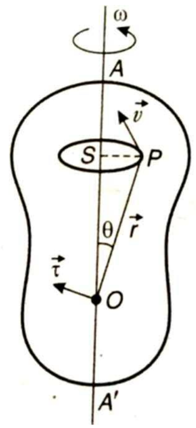
$$= \frac{2\omega_0\omega - 2\omega_0^2 + \omega^2 + \omega_0^2 - 2\omega \times \omega_0}{2\alpha}$$

$$\Rightarrow 2\alpha\theta = \omega^2 - \omega_0^2$$

$$\Rightarrow \omega^2 = \omega_0^2 + 2\alpha\theta$$

8. Derive the equation of motion for a rotation body.

Ans: Consider the case of a rigid body rotating with angular velocity ω about a fixed axis passing through point O. Every particle in the body moves in a circle with its center on the axis of rotation. Now consider a particle of mass *m* at P. Let its position vector be **r** with respect to O. The linear velocity of particle P will be $\bar{v} = \bar{\omega} \times \bar{r}$. Its direction is tangential to the circle at point P and perpendicular to **r**. The linear momentum of the particle is *m* **v**. The angular momentum is the moment of this linear momentum i.e., $\vec{l} = \bar{r} \times m\bar{v}$. The direction of $\vec{l}$ is perpendicular to both ***r*** and ***v***.



The total angular momentum of the whole body is given by

$$L = \Sigma \bar{l} = \Sigma m \bar{r} \times \bar{v}$$

$$= \Sigma\, m\bar{r} \times (\bar{\omega} \times \bar{r}) \;\ldots (1) \quad (\because v = \omega \times r)$$

Using the relation,

$a \times (b \times c) = (a \bullet c)\; b - (a \bullet b)c$, the equation can be written as

, $$L = \Sigma m\left[(r \bullet r)\vec{\omega} - (r \bullet \vec{\omega})r\right]$$

$$\Rightarrow L = \Sigma\, m\left[r^2\vec{\omega} - (r \bullet \vec{\omega})r\right] \qquad \ldots (2)$$

Let angle SOP $=\theta$. The component of **r** along the axis of rotation will be $r\cos\theta$. Hence component of L along the axis of rotation has a magnitude.

$$L_0 = \Sigma m\left[r^2\omega - r\omega\cos\theta \;\; r\cos\theta\right]$$

$$= \Sigma m\left[r^2\omega - r^2\omega\cos^2\theta\right]$$

$$= \Sigma\, mr^2\omega\left(1-\cos^2\theta\right) = \Sigma\, mr^2\omega\sin^2\theta$$

$$= \Sigma m r_0^2\omega$$

Where $r_0 = r\sin\theta$

here $r_0 =$ distance of the particle from axis of rotation.

$$\left(\because I = \Sigma m r_0^2\right)$$

$$\therefore\; L_0 = I\omega \qquad \dots (3)$$

Differentiating eq. (3) w.r.t. time, we get

$$\frac{dL_0}{dt} = I\frac{d\omega}{dt} \quad (\because I \text{ being constant}) \qquad \dots (4)$$

But $L_0 = \Sigma\, \overline{r}\times m\overline{v}$

$$\therefore\; \frac{dL_0}{dt} = \Sigma\, \overline{r}\times\frac{d}{dt}\left(m\,\overline{v}\right) = \Sigma\, \overline{r}\times\overline{F} = \overline{\tau} \qquad \dots (5)$$

($\because \overline{r}\times\overline{F}$ is the moment of force or torque)

Comparing eqs. (4) and (5), we get

$$\tau = I\frac{d\omega}{dt} \qquad \dots (6)$$

This is the equation of motion of rigid body.

9. *Explain about angular momentum and Moment of inertia tensor.*

Ans: The angular momentum for i[th] particle may be written as $\overline{L} = \sum m_i\left(\overline{r_i}\times\overline{v_i}\right) = \Sigma m_i\left[\overline{r_i}\times\left(\overline{\omega}\times\overline{r_i}\right)\right]$

$$\Rightarrow \overline{L} = \Sigma m_i\left[\left(\overline{r_i}\bullet\overline{r_i}\right)\overline{\omega} - \left(\overline{r_i}\bullet\overline{\omega}\right)\overline{r_i}\right] \qquad \dots (1)$$

Consider the general case for which $\overline{r_i}\bullet\overline{\omega}\neq 0$. For this consider the XYZ coordinate system fixed in the body. Now we consider the component form of eq. (1)

Here $\overline{r} = \overline{i}\,x + \overline{j}\,y + \overline{k}\,z$ and $\overline{\omega} = \overline{i}\,\omega_x + \overline{j}\,\omega_y + \overline{k}\,\omega_z$

$$\therefore \left(\overline{r_i}.\overline{r_i}\right) = \left(\overline{i}\,x + \overline{j}\,y + \overline{k}\,z\right).\left(\left(\overline{i}\,x + \overline{j}\,y + \overline{k}\,z\right)_i\right)$$

$$= \left(x_i^2 + y_i^2 + z_i^2\right)$$

$$\therefore L = \Sigma m_i\Big[\left(x_i^2+y_i^2+z_i^2\right)\left(\overline{i}\,\omega_x + \overline{j}\,\omega_y + \overline{k}\,\omega_z\right) -$$

$$\left(x_i\omega_x + y_i\omega_y + z_i\omega_z\right)\left(\overline{i}\,x + \overline{j}\,y + \overline{k}\,z_i\right)\Big]$$

$$= \Sigma m_i\Big[\left\{y_i^2+z_i^2)\omega_x - x_iy_i\omega_y - x_iz_i\omega_z\right\}i +$$

$$\left\{x_i^2+z_i^2)\omega_y - x_iy_i\omega_x - y_iz_i\omega_z\right\}j + \left\{x_i^2+y_i^2)\omega_z - x_iz_i\omega_x\right.$$

$$\left. - y_iz_i\omega_y\right\}k\Big] \qquad \dots (2)$$

We know that $\overline{L} = \overline{i}\,L_x + \overline{j}\,L_y + \overline{k}\,L_z$

Comparing eqs. (2) and (3), we get

$$L_x = \Sigma\left[m_i\left(y_i^2+z_i^2\right)\right]\omega_x + \left[-\Sigma m_ix_iy_i\right]\omega_y + \left[-\Sigma m_1x_iz_i\right]\omega_z$$

$$L_y = \Sigma\left[m_i\left(x_i^2+z_i^2\right)\right]\omega_y + \left[-\Sigma m_ix_iy_i\right]\omega_x + \left[-\Sigma m_1x_iz_i\right]\omega_z$$

$$L_z = \Sigma\left[m_i\left(x_i^2+y_i^2\right)\right]\omega_z + \left[-\Sigma m_ix_iy_i\right]\omega_x + \left[-\Sigma m_1x_iz_i\right]\omega_y$$

$$\dots (3)$$

The component of $\overline{L}$ in X direction involves three separate quantities. The quantities depend on the distribution of mass in the body and on instantaneous orientation of the direction of the angular velocity of the body relative to X, Y, Z axes. These quantities are known an inertial coefficients of the moment of inertia of a rotating body whose angular velocity vector changes with time. Thus

$$I_{xx} = \sum m_i \left(y_i^2 + z_i^2\right) \sum m_i \left(r_i^2 - x_i^2\right)$$

$$I_{xy} = -\sum m_i x_i y_i$$

$$I_{xx} = -\sum m_i x_i z_i$$

$$I_{xx} = -\sum m_i x_i z_i \qquad \ldots (4)$$

These are called inertial coefficients.

In terms of inertial coefficients eq. (3) can be written as

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

Similar $L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$ … (5)

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The matrix form of eq. (6) is given by

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \ldots (6)$$

The diagonal elements Ixx, Iyy and Izz are called as principal moment of inertia around X, Y and Z axes respectively. The other six terms i.e., $I_{xy}, I_{xz}, I_{yz}, I_{zx}$ and I_{zy} are called as off diagonal terms or products of inertia.

Eq. (6) can be expressed by using symbols 1, 2, 3 for x, y, z respectively.

Thus $L_\mu = \sum_{V=1}^{3} I_\mu v^{\omega V}$; $\mu = 1, 2$ and 3 … (7)

The more elegant vector form of eq. (7) is $L = \overleftrightarrow{I}\,\omega$

Where ω is the vector with three components $\omega_{x,}\omega_y$ and ω_z and $\overleftrightarrow{I}$ stands for an operator called as tensor.

10. *Derive Euler's equations of a rotating body.*

Ans: The torque $\vec{\tau} = \frac{d\bar{L}}{dt}$ … (1)

The motion of the rotating body may also be considered when the axes are attached to rotating body. As these axes are rotating with rotating body, this constitutes a non inertial frame with respect to the axes fixed in the space. We can transform the equations of motion of rotating body from body coordinates (non-inertial) to space coordinates(inertial).

$$\left(\frac{d}{dt}\ldots\right)_{space} = \left(\frac{d}{dt}\ldots\right)_{body} + \omega \times (\ldots)$$

i.e., $\left(\frac{d\bar{L}}{dt}\right)_{\mathbf{space}} = \left(\frac{d\bar{L}}{dt}\right)_{\mathbf{body}} + \omega \times \bar{L}$ … (2)

Where ω is the angular velocity of rotating frame.

From eqs. (1) and (2), $\bar{\tau} = \frac{d\bar{L}}{dt} + \bar{\omega} \times \bar{L}$ … (3)

Where L refers to the rotating frame.

If the body is symmetric, its axes of rotating coincide with the principal axes of symmetry. Denoting the body-axes as 1,2 and 3 instead of *x, y* and *z*, the rotating around 1-axis of symmetry as scalar can be expressed as

$$\tau_1 = \frac{dL}{dt} + \omega_2 L_3 - \omega_3 L_2 \qquad \ldots (4)$$

Because $\overline{\omega}\times\overline{L}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \omega_1 & \omega_2 & \omega_3\\ L_1 & L_2 & L_3\end{vmatrix}$

$$=\bar{i}\left(\omega_2 L_3-\omega_3 L_2\right)+\bar{j}\left(\omega_3 L_1-\omega_1 L_3\right)+\bar{k}\left(\omega_1 L_2-\omega_2 L_1\right)$$

From eq. (4)

$$\tau_1=I_1\frac{d\omega_1}{dt}+\omega_2 I_3\omega_3-\omega_3 I_2\omega_2 \quad \left(\because L=I\omega\right)$$

$$\Rightarrow \tau_1=I_1\frac{d\omega_1}{dt}+\left(I_3-I_2\right)\omega_2\omega_3 \qquad \dots (5)$$

Similarly, scalar component equations along 2-zxis and 3-axis can be written as

$$\tau_2=I_2\frac{d\omega_2}{dt}+\left(I_1-I_3\right)\omega_1\omega_3 \qquad \dots (6)$$

and $\tau_3=I_3\frac{d\omega_3}{dt}+\left(I_2-I_1\right)\omega_1\omega_2 \qquad \dots (7)$

Equations (5), (6) and (7) are known as Euler Equation of rotational motion for a right body fixed at one point.

Equations (5), (6) and (7) in terms of *x, y* and *z* axes can be expressed as

$$\tau_x=I_x\frac{d\omega_x}{dt}+\left(I_z-I_y\right)\omega_y\omega_z$$

$$\tau_y=I_y\frac{d\omega_y}{dt}+\left(I_x-I_z\right)\omega_x\omega_z$$

and $\tau_z=I_z\frac{d\omega_z}{dt}+\left(I_y-I_x\right)\omega_x\omega_y$

The above equations can be written in the following form for symmetry

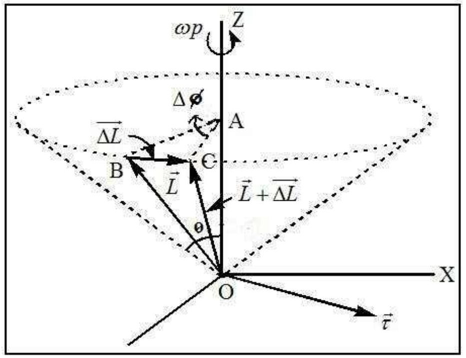
$$\tau_x=I_x\frac{d\omega_x}{dt}\left(I_z-I_y\right)\omega_y\omega_z$$

$$\tau_y=I_y\frac{d\omega_y}{dt}\left(I_x-I_z\right)\omega_x\omega_z$$

and $\tau_z=I_z\frac{d\omega_z}{dt}\left(I_y-I_x\right)\omega_x\omega_y$

11. Explain the precession of a spinning top. Derive an equation for processional velocity.

Ans: A symmetrical body rotating (or spinning) about an axis which is fixed at one point is called top. Fig. shows a top spinning about its axis of symmetry with angular velocity ω. The point O of this axis is fixed and is taken as the origin of an inertial reference frame. We know that the axis of spinning top moves around the vertical axis OZ, sweeping out a cone. The rotation of the axis of rotation of the spinning top is called precession. The axis about with the direction of rotation of the body processes is called the axis of precession.



Let at any instant the axis of the tope makes on angle θ with the vertical axis OZ and its angular momentum is L.

Torque and acceleration: Let C can be center of mass of the top with position vector **r** with respect to O. The weight of the top ***mg*** acts on the top vertically downwards from C due to earth's gravitational pull. So the torque exerted by the force ***mg*** on the top about O can be written as ,

$$\bar{\tau} = \bar{r} \times \bar{F} = \bar{r} \times m\bar{g}$$

The scalar magnitude is given by

$$\tau = r\,mg\sin\left(180^0 - \theta\right) = r\ mg\sin\theta \quad \dots (1)$$

And its direction is perpendicular to the plane containing **r** and ***mg*** obtained by right hand rule. This is shown in fig. Thus the torque τ is perpendicular to **L** i.e., perpendicular to the axis of rotation of the top.

The torque produces an angular acceleration α in its direction (as = $\tau = 1\alpha$) i.e, in direction perpendicular to ω. Due to this angular acceleration α, ω changes in direction but not in magnitude. So the axis of top and hence ω, **L** and **r** all processes about OZ.

The angular momentum of body is conserved only when the external torque acting on it is zero. Here a torque acts on the top and hence its angular momentum will change. The change will take place in the direction of the torque i.e., perpendicular to **L.** Let in a short time Δt the torque τ produce a change ΔL in angular momentum. Then

$$\bar{\tau} = \frac{\Delta \bar{L}}{\Delta t} \Rightarrow \Delta \bar{L} = \tau\ \Delta t \quad \dots (2)$$

After a time Δt, the angular momentum is $L + \Delta L$. Since ΔL is perpendicular to **L** and is assumed to be very small in magnitude compared to **L,** the new angular momentum $L + \Delta L$ has a magnitude $\approx$ L but in different directions. This means that angular momentum remains constant in magnitude but varies in direction. As the torque vector completes a circle in X-Y plane, the vector L

completes a cone in space with its axis as the Z-axis of the inertial frame. In this way vector **L** always lies along the axis of rotation of the top, the axis of rotation itself rotates about a vertical axis (Z-axis), and sweeps out a cone whose vertex is the fixed point O. The motion of the axis of top is called precession of the top.

Processional Velocity : $\left(\omega_p\right)$

The angular speed ω_p with which the top precesses about the Z-axis can be calculated as follows :

Let $\Delta\phi$ be the angle turned by the head of angular momentum vector **L** in the time Δt, the rate of precession.

$$\omega_p = \frac{\Delta\phi}{\Delta t}$$

From fig. $\Delta\phi = \frac{BC}{AC} = \frac{\Delta L}{OC\sin\theta} = \frac{\Delta L}{L\sin\theta}$

$$\left(\because OC = L + \Delta L \approx L\right)$$

$$\therefore \quad \omega_p = \frac{\Delta L}{L\sin\theta.\Delta t} = \frac{\tau}{L\sin\theta} \quad \left(\because \Delta L / \Delta t = \tau\right)$$

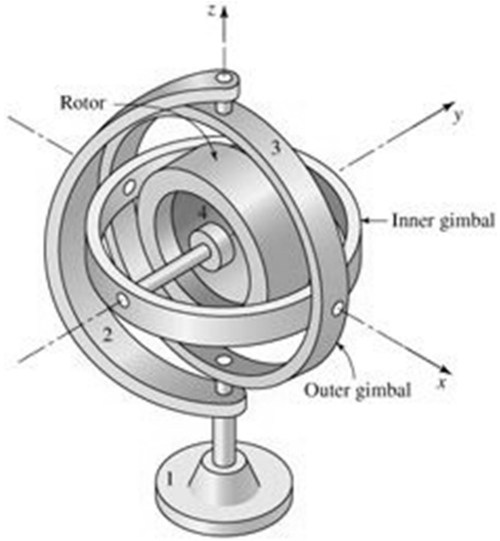
Substituting the value of magnitude of τ from eq. (1), we get

$$\omega_p = \frac{m\,g\,r\sin\theta}{L\sin\theta} = \frac{m\,g\,r}{L} = \frac{m\,g\,r}{I\omega} \quad \dots (3)$$

12. Explain the working of a Gyroscope.

Ans: Gyroscope : If the fixed point, about which a symmetrical body the body, then it is known as gyroscope. The gyroscope is shown in fig.

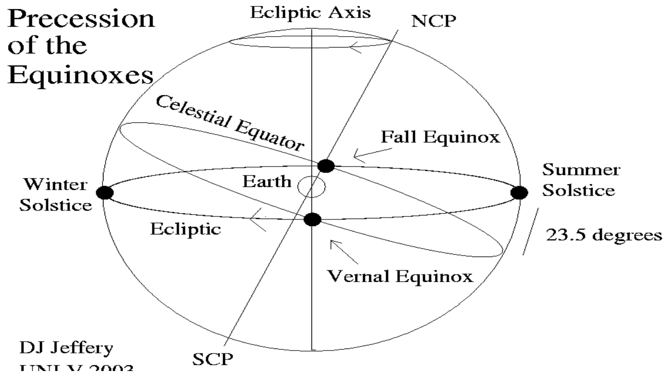
The gyroscope consists of a large, heavy wheel whose moment of inertia is very large. The wheel can rotate freely in any direction about an axle passing through its center of



mass. This is achieved by double pivot provided in two circular rings R_1 and R_2 placed at the top of a stand F. In this way the wheel and axle can turn freely about any one of the mutually perpendicular axes. When the wheel is rotating at a high speed, the axis of rotation remains the same relative to the frame even it the orientation of the frame F is changed. If however, any torque is applied perpendicular to the axis of rotation, there will be a precession of the axis. A change in the rate of precession causes the spin axis of the wheel to rise or fall. Now the spin axis starts oscillating up and down about its mean position. This type of motion is called nutation. We known that the rate of precession is inversely proportional to the angular momentum $(I\omega)$ of the wheel and the wheel has large moment of inertia, so it suffers a very small precession. In this way a greater stability of the axis of rotation can be achieved by increasing the moment of inertia of the wheel and the speed of rotation. The gyroscope is a device characterized by greater stability of its axis of rotation.

13. Explain the Precession of the equinoxes

Ans: We know that equatorial plane of the earth and the plane of its orbit round the Sun are inclined to each other at an angle 23.5^0. The two planes intersect at points A and B as shown in fig. Point A is known as vernal equinox while point B is known as autumnal equinox. The earth in one complete round about the sun crosses point A (vernal equinox) at about 21st March and point B (autumnal equinox) at about 22nd September. The line joining the points A and B is known as line of equinoxes. At equinox day and night are equal.



We know that the earth is not a perfect sphere but bulges out at the equator. Further the gravitational attraction due to the sun and the moon on the equatorial bulge of the earth give rise to a torque as the two forces are not equal this torque makes the axis of earth to process. As the earth acts like a top, its precessing axis describes a circular cone relative to the pole star assumed to be fixed. This precessional motion of the earth's axis causes a change in the direction of the line of equinoxes which is called the precession of equinoxes.

14. Explain the Precession of atom and nucleus in magnetic field.

Ans: Many atomic and nuclear systems with a magnetic moment also have angular momentum and effective internal electric current proportional to their angular momentum. Nuclei precess around the magnetic field for essentially the same reasons that tops or gyroscopes precess around a gravitational field. **Larmor's precession** is the precession of the magnetic moment of an object about an external magnetic field

1) Both gyroscopes and nuclei possess angular momentum. For the gyroscope, angular momentum results from a flywheel rotating about its axis. For the nucleus, angular momentum results from an intrinsic quantum property (spin).
2) Momentum is also sometimes called inertia. Objects possessing momentum have a tendency to maintain their motion unless acted upon by an external force. For example, a speeding truck has a great deal of (linear) momentum and cannot easily be induced to change its speed or direction. Angular momentum behaves similarly, conferring on the nucleus or gyroscope a strong resistance to changing its orientation or direction of rotation.
3) Static gravitational and magnetic fields create a torque or "twisting force" acting perpendicular to both the field and the direction of the angular momentum. The gyroscope or nucleus does not "tip over" but is instead deflected into a circular path perpendicular to the field.
4) The resultant circular motion is called precession. Precession occurs at a specific frequency denoted either by ω_o (called the angular frequency) or f_o (called the cyclic frequency). $\omega_0 = 2\pi f_0$
5) The precession frequency of a gyroscope is a function of the mass and shape of the wheel, the speed of wheel rotation, and the strength of the gravitational field. The precession frequency of a nucleus is proportional to the strength of the magnetic field (Bo) and the gyromagnetic ratio (γ), a particle-specific constant incorporating size, mass, and spin. Then the Larmor's relationship, given by the equation: $f_0 =$ Y Bo

SHORT ANSWER QUESTIONS

15. State the Newton's laws of motion.

Ans:* *First law:

Every body continues in its state of rest or of uniform motion in a straight line unless an external force acts on it to change that state.

Second law:

The rate of change of momentum of a particle is equal to the force acting on it and takes place in the direction of force.

Third law:

To every action there is always an equal and opposite reaction.

16. what is a variable mass system. Explain the motion of a rocket

Ans: If the mass of the system varies with time, then it is called variable mass system.

Ex: motion of a rocket.

The motion of rocket is an example of system of variable mass. The rocket consists of a combustion chamber in which liquid or solid fuel is burnt. When the fuel is burnt, the pressure inside the combustion chamber rises very high. Due to this high pressure hot gasses are expelled from the tail of the rocket. Then according to the law of conservation of momentum, the rocket moves in the direction opposite to the direction of hot gas jet. As the mass of the fuel inside the rocket decreases with time, the velocity of the rocket increases.

The velocity of the rocket at any instant of time t is given by, $\therefore V = V_0 + u\log\frac{M_0}{M} - gt$

When the rocket starts from rest, V_0=0,

$$\therefore V = u\log\frac{M_0}{M} - gt$$

Ignoring the gravity effect, $V = u\log\frac{M_0}{M}$.

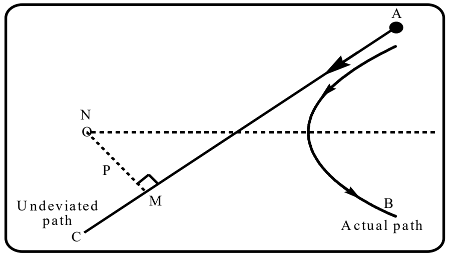
17. *Explain the working of a multistage rocket.*

Ans: The working of the rocket depends on the law of conservation of momentum. According to the law of conservation of momentum, the gas jet emerging in the backward direction makes the rocket to move in the forward direction. The velocity attained by the rocket is nearly 4 km/sec.. To obtain higher velocities, multistage rockets containing two or more stages are used the first stage of the rocket is used to acquire the acceleration of the rocket. When the fuel of the first stage is exhausted, it detaches from rocket and drops off. The velocity at this stage becomes the initial

velocity of the second stage. Now the second stage starts functioning. The rocket gains acceleration and its velocity goes on increasing. The removal of the excess mass contained in the first stage considerably helps in attaining the higher velocity. When the fuel of the second stage is exhausted, it also detaches from the rocket. The velocity acquired so far by rocket is less than the escape velocity (11.2 km/sec.). Finally the third stage of the rocket starts with the required velocity.

18. *Explain the terms impact parameter, scattering cross-section.*

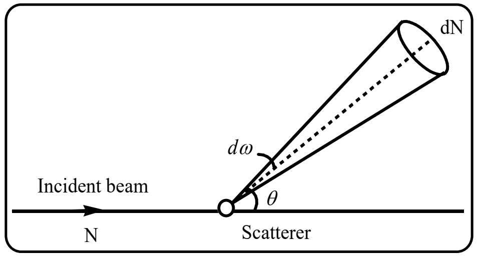
Ans: Impact Parameter: Consider a positive particle, like a proton or an $\alpha-$particle, approaching a massive nucleus N of an atom, as shown in fig. Due to Coulomb force of repulsion, the particles follow a hyperbolic path AB with nucleus N as its focus. In the absence of the repulsive force, the particle would have followed the straight-line path AC. As shown in figure, p is the perpendicular distance from the nucleus N to the original direction AC of the particle. The distance NM = p is called the impact parameter.



Thus impact parameter is defined as the closest distance between nucleus and positively charged particle projected towards it. This is also known as collision parameter.

Scattering Cross-Section: When $\alpha-$particles are projected into a thin metal foil, they are deflected or scattered in different directions. Let N be the incident intensity (number of incident particles crossing per unit time a unit surface placed perpendicular to the direction of propagation).

Suppose dN be the number of particles scattered per unit time into solid angle $d\omega$ located in the direction θ and ϕ with respect to the bombarding direction.



The ratio $\frac{dN}{N}$ is called scattering cross section.

Thus the scattering cross section in a given direction is defined as the ratio of number of scattered particles into solid angle $d\omega$ per unit time to the incident intensity.

19. Define rigid body. Mansion rotational kinematics relations.

Ans: *A body which does not undergo any change in shape or size by the application of external forces is called rigid body.*

A rigid body can be defined as a solid and fixed. The distance between the particles is not disturbed by any external forces applied.

Rotational Kinematics Relations:

I. $\omega = \omega_o + \alpha t$

II. $\theta = \omega_o t + \frac{1}{2}\alpha t^2$

III $\omega^2 = \omega_0^2 + 2\alpha\theta$

20. Give Euler's equations of a rotating body.

Ans:

$$\tau_x = I_x \frac{d\omega_x}{dt}\left(I_z - I_y\right)\omega_y\omega_z$$

$$\tau_y = I_y \frac{d\omega_y}{dt}\left(I_x - I_z\right)\omega_x\omega_z$$

and $$\tau_z = I_z \frac{d\omega_z}{dt}\left(I_y - I_x\right)\omega_x\omega_y$$

These are the ***Euler's equations of a rotating body***

21. Explain the precession of a spinning top.

Ans: A symmetrical body rotating (or spinning) about an axis which is fixed at one point is called top. Fig. shows a top spinning about its axis of symmetry with angular velocity ω. The point O of this axis is fixed and is taken as the origin of an inertial reference frame. We know that the axis of spinning top moves around the vertical axis OZ, sweeping out a cone. The rotation of the axis of rotation of the spinning top is called precession. The axis about with the direction of rotation of the body processes is called the axis of precession.

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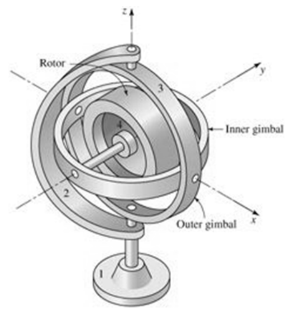

The gyroscope consists of a large, heavy wheel whose moment of inertia is very large. The wheel can rotate freely in any direction about an axle passing through its center of mass. This is achieved by double pivot provided in two circular rings R_1 and R_2 placed at the top of a stand F. In this way the wheel and axle can turn freely about any one of the mutually perpendicular axes. When the wheel is rotating at a high speed, the axis of rotation remains the same relative to the frame even it the orientation of the frame F is changed. If however, any torque is applied perpendicular to the axis of rotation, there will be a precession of the axis. A change in the rate of precession causes the spin axis of the wheel to rise or fall. Now the spin axis starts oscillating up and down about its mean position. This type of motion is called nutation. We known that the rate of precession is inversely proportional to the angular momentum $(I\omega)$ of the wheel and the wheel has large moment of inertia, so it suffers a very small precession. In this way a greater stability of the axis of rotation can be achieved by increasing the moment of inertia of the wheel and the speed of rotation. The gyroscope is a device characterized by greater stability of its axis of rotation.

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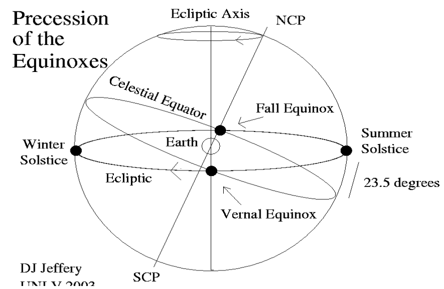

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SOLVED PROBLEMS

24. *A rocket burns 0.02 kg of fuel per second ejecting it as a gas with a velocity of 10,000 m/sec. What force does the gas exert on the rocket ?*

Sol: The thrust ($F_{reaction}$) exerted by the escaping gas on the rocket is given by $F_{reaction} = u\frac{dM}{dt}$

Here u = 10,000 m/sec and $\frac{dM}{dt}$ = 0.02 kg.

$F_{reaction}$ = (10,000) (0.02) = 200 N

25. *A rocket of mass 20kg has 180 kg fuel. The exhaust velocity of the fuel is 1.6 km/sec. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground.*

Sol: Given mass of the rocket = 20 + 180 = 200 kg and

u $u = 1.6 \times 1000\ m/s$

The rocket may rise from the ground when the thrust on the rocket is equal to its own weight, i.e.,

$$u = \frac{dM}{dt} = Mg$$

$$\Rightarrow \quad \frac{dM}{dt} = \frac{Mg}{u} = \frac{200 \times 9.8}{1.6 \times 10^3}$$

$$\Rightarrow \quad \frac{dM}{dt} = 1.225 \text{ kg/sec.}$$

∴ Rate of consumption of the fuel = 1.225 kg/sec.

26. *In the case of rocket motion, shown that greater the value of M_{fuel} / $M_{vehicle}$ the greater be speed attained by rocked.*

Sol: We known that $v = v_0 + u\ \log_e \frac{M_0}{M}$.

Where V = velocity of the rocket at any instant,

V_0 = initial velocity of rocket.

M_0 = (initial mass of rocket + fuel),

M = mass of the rocket at any instant and

u = exhaust velocity of gases.

When the fuel is burnt out completely, the remaining mass corresponds to the mass of vehicle (M_v). Above equation now can be written as

$$V = V_0 + u \log_e \frac{M_0}{M}$$

$$= V_0 + u \log_e \left[1 + \frac{M_0}{M_v} - 1\right]$$

$$= V = V_0 + u \log_e \left(1 + \frac{M_0 - M_V}{M_V}\right)$$

$$= V_0 + u \log_e \left(1 + \frac{M_{fuel}}{M_{vehicle}}\right)$$

So, greater is the value of $\frac{M_{fuel}}{M_{vehicle}}$, the greater is the speed attained by the rocket.

27. *A rocket having an initial mass M_0 starts from rest. When it attains a velocity v, it mass becomes M. What is the ratio of $\frac{M_0}{M}$ when the velocity of exhaust gases is equal to υ (numerically).*

Sol: The velocity V of the rocket at any instant of time t is given by

$$v = v_0 = u \log_e \left(\frac{M_0}{M}\right)$$

Given that $V_0 = 0, u = V$

$$V = 0 + \text{V} \log_e \left(\frac{M_0}{M}\right)$$

$$\Rightarrow \quad \log_e \left(\frac{M_0}{M}\right) = 1 \Rightarrow \frac{M_0}{M} = 2.717$$

28. *An empty rocket weights 5000 kg and contains 40,000 kg of fuel, if the exhaust velocity of the fuel is 2.0 km/sec. Find the maximum velocity gained by the rocket. (Given that log, 10 = 2.3, log_{10} 3 = 0.4771.*

Sol: Ignoring gravity effect, the velocity V of a rocket at anytime t is given by

$$V = V_0 + u\log_e \frac{M_0}{M}$$

According to the given problem

$V_0 = 0, M_0 5000 + 40,000 = 45,000kg.$

M = 5000 kg and u = 2.0 km/sec.

$$V_{max} = (2)\log_e \frac{45000}{5000}$$

$$= (2)\log e(3)^2 = (2)2\log_e 3$$

$$= (2)2 \times 2.3 \times 0.4771$$

= 4.4 km/sec.

29. *Suppose an engine of a rocket has to achieve a thrust of 3.3×10⁷N. At what rate must the fuel be consumed? Assume that the exhaust velocity of the hot gas from engine is 2900 m/s.*

Sol: Given, force = 3.3×10^7N

We know that

$$F = u\frac{dM}{dt} \text{ or } \frac{dM}{dt} = \frac{F}{u}$$

$$\frac{dM}{dt} = \frac{3.3 \times 10^7}{2900} = \frac{3.3 \times 10^5}{29} = \frac{33 \times 10^4}{29}$$

$$= \mathbf{1.1 \times 10^4 \text{ kg/sec}}$$

30. *A rocket starts from rest with an initial mass M_0 and its mass at burnt out is M. Neglecting all external forces, find the ratio of (M_0 / M) if the rocket speed is twice the exhaust speed.*

Sol:

$$\upsilon = u\log_e\left(\frac{M_0}{M}\right)$$

Given that, $\upsilon = 2u$

$$\therefore \quad 2u = u\log_e\left(\frac{M_0}{M}\right)$$

$$\Rightarrow 2 = \log_e\left(\frac{M_0}{M}\right)$$

$$\Rightarrow \frac{M_0}{M} = e^2$$

31. *A rocket having initial mass 240kg ejects fuel at the rate of 6 kg/s with a velocity 2 km/s vertically downward relative to itself. Calculate its velocity 25 seconds after start, taking initial velocity to be zero and neglecting gravity.*

Sol: Mass consumed in 25s = 6 × 25 = 150 kg

Remaining mass M = 240 – 150 = 90 kg.

Now, $\upsilon = \upsilon_0 + u\log_e\left(\frac{M_0}{M}\right)$

$$= 0 + 2 \times 2.3 \times \log_{10}\left(\frac{240}{90}\right) = 4.6\log_{10} 2.67$$

$$= 4.6 \times 0.4265 = 1.9619 \text{ km/sec}$$

32. *A rocket of mass 200 kg has 1800 kg of fuel. The exhaust velocity of the fuel is 2.45 km/sec. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground. Also calculate the ultimate vertical speed gained by the rocket, when the rate of consumption of the fuel is 20 kg/sec.*

Sol: For the minimum rate of consumption of fuel so that the rocket may rise from the ground ,

Magnitude of thrust = Weight of rocket

$$\Rightarrow \quad u\frac{dM}{dt} = Mg$$

$$\Rightarrow \quad \frac{dM}{dt} = \frac{Mg}{u} = \frac{200\times 9.8}{2.45\times 1000} = 0.8 \text{ kg/sec}$$

The time for consumption of all fuel = $\frac{1800}{20} = 90$ s

since the rate of consumption of fuel is 20 kg/sec. Now.

$$\upsilon_{\max} = u\log_e\frac{M_0}{M} - g\,t$$

$$= \left(2.45\times 10^3\right)\log_e\frac{2000}{200} - 9.8\times 90$$

$$= \left(2.45\times 10^3\right)2.303 - 882 = 5642.4 - 882$$

$$= 4760.4 \text{ m/s} = 4.76 \text{ km/s}$$

33. *A rocket burns 0.05 kg of fuel per second and ejects the burnt gases with a velocity of 5000 m/s. Find the reaction.*

Sol: Reaction = $\frac{\text{dM}}{\text{dt}}\times\upsilon_{rel.}$

$$= 0.05\times 5000 = 250 \textbf{ N}$$

$$t = \frac{360}{4} = 90\,\text{sec}.$$

Now $V_{\max} = V_0 + u\log_e\left(M_0/M\right) - gt$

$$= 0 + 2\log_e\left(\frac{400}{40}\right) - \left(\frac{9.8\times 90}{1000}\right)$$

$$= \left[2\times 2.303\,\log_{10}10\right] - 0.882$$

$$= 3.724 \text{ kg/sec.}$$

34. *The stage of two stage rocket separately weight 100 and 10 kg and contain 800 kg and 90 kg fuel respectively. What is the final velocity that can be achieved with an exhaust velocity of 1.5 km/sec. ?*

Sol: We know that $V_1 = V_0 + u\log_e\left(\frac{M_0}{M}\right)$

For first stage, $M_0 = 800 + 90 + 100 + 10 = 1000kg$

$$M = 90 + 10 + 100 = 200kg$$

$V_0 = 0$ and u = 1.5 km/sec.

$$V_1 = 0 + 1.5\,\log_e\left(\frac{1000}{200}\right) = 1.5\log_e(5)$$

$$= 1.5\times 2.3\,\log_{10}5 = 2.415 \text{ km/sec.}$$

This will be the initial velocity for the second stage.

For second stage, $M_0 = 100kg$, M = 10kg,

$V_0 = V_1 = 2.415$ km/sec. Thus

$$V_2 = V_1 + u\log_e\left(\frac{M_0}{M}\right)$$

$$= 2.415 + 1.5\log_e\left(\frac{100}{10}\right)$$

$$= 2.415 + 1.5 \times 2.3\log_{10}(10) = 5.865 \text{ km/sec.}$$

So, the final velocity achieved by the rocket = 5.865 km/sec.

35. A 6000 kg. rocket is set for vertical furing. If the exhaust speed is 1000 m/sec, how much gas must be ejected each second to supply the thrust needed (a) to overcome the weight of the rocket, (b) to give the rocket an initial upward acceleration of 20 m/sec² ?

Sol: a) To overcome the weight of rocket,

Magnitude of thrust = weight of the rocket

$$u\frac{dM}{dt} = Mg$$

$$\Rightarrow \frac{dM}{dt} = \frac{Mg}{w} = \frac{6000 \times 9.8}{1000} = 58.8 \text{ kg/sec.}$$

Hence the rate of consumption of fuel is 58.8 kg/sec.

b) To given an initial upward acceleration of 19.6 m/sec², the thrust on the rocket should be equal to the weight of the rocket plus the force required to produce the acceleration of 19.6 m/sec².

Hence, Magnitude of thrust = weight of rocket + M a

$$\Rightarrow u\frac{dM}{dt} = Mg + Ma$$

$$\Rightarrow \frac{dM}{dt} = \frac{M(g+a)}{u} = \frac{6000(9.8+20)}{1000} = 178.8 \ Kg/sec.$$

36. A couple of 10^8 dynes cm is applied to a fly wheel of mass 10kg and radius of gyration 50 cm. What is the resulting angular acceleration ?

Sol: We know that

$$C = I\alpha = I\frac{d\omega}{dt} \text{ and } I = MK^2$$

$$\therefore 10^8 = (10 \times 1000 \times 50^2)(d\omega/dt)$$

$$\Rightarrow \frac{d\omega}{dt} = \frac{10^8}{10^4 \times 50^2} = 4 \text{ radians/sec}^2$$

$\therefore$ angular acceleration = 4 rad/sec²

37. The speed ofa particle moving along a circle of radius 20 cm increases at the rate of 10 cm/sec². If the mass is 200 gm, find the torque acting on it.

Sol: The angular momentum L is given by L = mvr and $\tau = \frac{dL}{dt} = mr\frac{dv}{dt}$.

Given m = 200 gm = 0.2 kg; r = 20 cm = 0.2 m and

$$\frac{dv}{dt} = 10cm/\text{sec}^2 = 0.1 \text{ m/sec}^2$$

$$\therefore \tau = 0.2 \times 0.2 \times 0.1 = 0.004 \quad N-m$$

38. A sphere of mass 2.5kg and radius 0.5m is revolving without slipping on a horizontal road with a velocity of 2 ms⁻¹. Calculate its K.E. of motion.

Sol: When a sphere rolls on a plane surface, without slipping, it has translational as well as rotational K.E.

K.E. of translation = $\frac{1}{2}mv^2 = \frac{1}{2} \times 2.5 \times 2^2 = 5$ joules

K.E of rotation = $\frac{1}{2}I\omega^2 = \frac{1}{2}\times\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 \ (\because v = r\omega)$

$$= \frac{1}{5}mv^2 = \frac{1}{5}\times 2.5\times 2^2 = 2 \text{ joules}$$

Total K.E = 5 + 2 = 7J

39. A fly wheel of mass 500 kg and diameter 1m changes its angular frequency from 0 to 18 revolutions per minute during 5 seconds. Find the torque.

Sol: The moment of inertial 1 of the fly wheel is given by,

$I = MK^2$ and $K = R = 0.5m$

$\therefore I = 500(0.5)^2 = 125 kg\, m^2$

Initial angular velocity = 0

Final angular frequency in 5 sec = 18 r.p.m.

$\therefore$ final angular velocity = $\frac{18\times 2\pi}{5} = 7.2\pi \ rad/\sec$

Hence change in angular velocity in 5 sec. $= 7.2\ \pi - 0$

$= 7.2\pi \ rad/\sec$

Angular acceleration $(\alpha) = \frac{\text{change in angular vel}}{\text{time}} = \frac{7.2\pi}{0.5}$

$= 1.44\ \pi \ rad/\sec^2$

The torque (τ) given by

$\tau = I\alpha = 125\times 1.44 \quad p = 565.4 \ N-m$.

40. A 20 kg object is accelerating on the circumference of a circle of radius 1.5m. If the rate of increase in velocity is 0.5 m/s, find the torque acting on it.

Sol: The angular momentum L is given by L = mvr

and $\tau = \frac{dL}{dt} = mr\frac{dv}{dt}$

Given m = 20 kg; r = 1.5m and dv/dt = 0.5 m/sec^2

$\therefore \tau = 20\times 1.5\times 0.5 = 1.5 \ N-m.$

41. A solid sphere of diameter 2 cm. having a moment of inertia 2x 10^{-6} kg. m^2 about its diameter rolls without slipping with a velocity 5cm/ sec. Calculate the K.E. of rotation.

Sol: K.E of rotation = $\frac{1}{2}I\omega^2$ and v = $r\omega$

$$= \frac{1}{2}\times 2\times 10^{-6}\times\frac{v^2}{r^2} = 10^{-6}\left(\frac{5\times 100}{1\times 100}\right)^2$$

$$= 10^{-6}\times 5^2 = 25\times 10^{-6} \ joule$$

42. A circular disc of 50 kg and radius 100 cm is mounted axially and made to rotate. Calculate the K.E. it possesses when executing 100 rotations per minute.

Sol: No. of rotations per min = 100

No. of rotations per sec (n) = 100 / 60 = 5/3

$\therefore$ Angular velocity (ω) = $2\pi n = 2\pi(5/3)$

$= 10\pi/3 rad/\sec$

The moment of inertia I of the circular disc passing through its centre and perpendicular to its plane is given by

$$I = \frac{MR^2}{2} = \frac{50\times I^2}{2} = 1250 kg - m^2$$

$\therefore$ Kinetic energy of circular disc $= \frac{1}{2}I\omega^2 = \frac{1}{2}\times 1250\left(\frac{10\pi}{3}\right)^2$ = 70080 joules

43. *A car engine develops 75 KW power when rotating at a speed of 100 rpm. What is the torque acting ?*

Sol: The power developed by the torque τ exerted on a rotating body is given by

$P = \tau\omega$

$\Rightarrow \quad \tau = P/\omega$

$P = 75KW = 75000$ watt

$\omega = (1000/60)2\pi$

$= (100/3)\pi$ rad/sec.

$$\therefore \tau = \frac{75000}{(100/3)\pi} = \frac{750\times3}{\pi} = 716.3 \text{ joule}$$

44. *Find the magnitude of the angular momentum of a cycle wheel of mass 2 kg and radius 0.5 m when rolling at a speed of 24 km ph.*

Sol: Given, M = 2 Kg, r = 0.5 m, V = $24\times\frac{5}{8} = \left(\frac{120}{18}\right) m/\text{sec}$.

L = ?

But $L = I\omega = Mr^2\times\frac{v}{r} = Mrv \ (\therefore v = r\omega)$

$$\therefore L = \left(\frac{2\times0.5\times120}{18}\right) = 6.667 \text{ kg.m}^2 \text{ rad/sec}$$

45. *Find the orbital angular momentum of moon about the earth, given the mass of the moon = 7.36 × 10²³ kg, average distance of the moon from the earth = 3.84 × 10⁸ m, and period of revolution of the moon = 27.3 days.*

Sol: Angular velocity of moon w is given by, $\omega = \frac{2\pi}{t}$

$$\Rightarrow \omega = \frac{2\pi}{27.3\times24\times60\times60} = 2.663\times10^{-6} \text{ rad/sec.}$$

Angular momentum of moon of mass m about the earth is given by

$L = r\times(mv = |m(r\times v)| = mvr\sin\theta$

As the moon is revolving round the earth in circular orbit, $\theta = 90^0$ and $\sin\theta = 1$.

$L = mvr$ and v = rω

$= mr^2\omega$

Given $m = 7.36\times10^{23}\,kg; r = 3.84\times10^8\,m$ and

$\omega = 2.663\times10^{-6}$ rad/sec.

Hence $L = (7.36\ 10^{23})\times(3.84\times10^8)^2\times(2.663\times10^{-6})$

$= 2.889\times10^{35}\ joule-\text{sec}$

46. *A fly-wheel of mass 10 kg. and radius of gyration 0.25 m makes 3 revolutions per second. Find its K.E.*

Sol: The moment of inertia of the fly-wheel is given by

$I = MK^2 = 10\times0.25^2 = 0.625$ kg m^2

and $\omega = 2\pi$ n/t = $2\pi\times3/1 = 6\pi$ radians/sec.

The K.E. of the fly-wheel is given by

$$K.E = \frac{1}{2}I\omega^2 = \frac{1}{2}\times0.625\times(6\pi)^2 = 177.6 \text{ joules}$$

47. *A grind-stone has a moment of inertia 6×10^7 gm.cm² about its axis. A constant couple is applied to the grind-stone is found to have a speed of 150 r.p.m in 10 sec. after starting from rest Calculate the couple.*

Sol: Initial angular velocity = 0

Final angular velocity in 10 sec. = 150 r.p.m.

$$=\frac{150\times 2\pi}{60}5\pi \text{ rad/sec}$$

$\therefore$ Change of angular velocity in 10 sec.

$= 5\pi - 0 = 5\pi$ rad/sec.

i.e., Angular acc = $\frac{\text{change in angular vel}}{\text{time}}=\frac{5\pi}{10}=\frac{\pi}{2}$ rad/sec^2

But couple = moment of inertia x angular acc.

$\therefore$ Couple = $6\times 10^7\times\frac{1}{2}\pi = 9.423\times 10^7$ dyne.cm

48. A disc of mass 2 kg rolls without slipping over a horizontal plane with a velocity of 4 ms^{-1}. Find the kinetic energy of the disc.

Sol: When a circular disc rolls on a plane surface, without slipping, it has kinetic energy of rotation as well as translation. Hence

Total K.E = $\frac{1}{2}I\omega^2+\frac{1}{2}Mv^2$

$$=\frac{1}{2}\left[\frac{1}{2}Mr^2\right]\omega^2+\frac{1}{2}Mv^2 \text{ and } \omega = \text{v/r}$$

$$=\frac{1}{4}Mr^2\left(v/r\right)^2+\frac{1}{2}Mv^2=\frac{1}{4}Mv^2+\frac{1}{2}Mv^2$$

$$=\frac{3}{4}Mv^2=\frac{3}{4}\times 2\times 4^2=24 \text{ joule}$$

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UNIT-II

3. MOTION IN A CENTRAL FORCE FIELD

ESSAY QUESTIONS

1. *Define a central force with examples. Mention the properties of central forces.*

Ans: Definition :

A central force is defined as a force which always acts on a particle towards or away from a fixed point and whose magnitude depends only on the distance from the fixed point. This fixed point is known as the center of the force.

Let O be the center of force, which is taken as the origin of coordinate system. P is a particle whose polar coordinates are r and θ. The central force on particle P is expressed by F. Mathematically. F can be expressed as

$$F = \hat{r} f(r) \qquad \dots (1)$$

Where *f(r)* is a function of the distance *r* of the particle from the fixed point and $\hat{r}$ is a unit vector along the radius vector *r* of the particle with respect to that fixed point. In case of two particles the magnitude of central force depends upon the distance of separation of two particles and the direction being along the line joining the particles.

Examples: (1) The gravitational force exerted on a particle by another particle which is stationary in an inertial frame of reference s a central force.

$$F = \frac{m_1 m_2}{r^2} = \hat{r} \quad \Rightarrow F \alpha \frac{1}{r^2}$$

Negative sign indicates that force is attractive. The earth moves around the sun under a central force which is always directed towards the sun.

(2) The electrostatic force exerted on a charged particle by another stationary, charged particle is a central force. The electrostatic force between two charges q_1 and q_2 separated at a distance *r* is given by

$$F = \frac{1}{4\pi \in_0} . \frac{q_1 q_2}{r^2} \qquad \Rightarrow F \alpha \frac{1}{r^2}$$

The electron in hydrogen atom moves under a central force which is always directed towards the nucleus.

(3) A particle attached to one end of a spring whose other end is stationary in an inertial frame of reference is an example of central force. The elastic force acting on the mass is expressed as $F = -K x$.

Where *x* is the distance of the mass from the unstretched position of the spring and K is spring constant.

Properties of Central Force :

(i) The general form of central force is represented by $F = \hat{r} f(r)$, where *f(r)* is a function of distance *r* of the particle from the fixed point and *r* is a unit vector along the radius vector *r* of the particle with respect to that fixed point.

(ii) Central force is a conservation force i.e., the work done by the force in moving a particle from one point to another is independent of the path followed.

(iii) Under a central force, the torque acting on the particle is always zero.

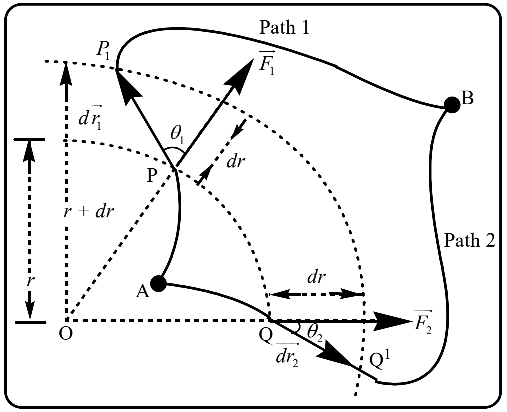
(iv) Under a central force, the angular momentum of the particle remains conserved.

(v) Under a central force, the areal velocity of the particle remains constant.

(vi) The central force is attractive when $f(r) < 0$ i.e., negative and repulsive when $f(r) > 0$ i.e., positive.

2. *What is a conservative force? Show that the central force is conservative in nature.*

Ans: Aforce is said to be conservative when the work done by the force in moving a particle from a point A to a point B is independent of the path followed between A and B. Thus the work done by a conservative force between the points A and B is the same for all the paths. The work done depends only on the particle's initial and final position. The work done by a conservative force along a closed path is zero.



To prove that the Central force is a conservative force :

Consider two points A and B connected by two arbitrary paths 1 and 2, as shown fig. Let a particle moves from point A to point B along any paths under a central force

which is directed away from a point O. Taking O as the centre, draw to arcs of radii r and $r + dr$ respectively. These are shown by dotted lines in fig. These arcs of the circles cut the paths 1 and 2 at P, P^1 and Q, Q^1 respectively, as shown. Let dr_1 and dr_2 be the displacement of the particle between the arcs along path 1 and path 2 respectively. Let F_1 and F_2 be the central forces acting on the particle at points P and Q. By the definition of central force,F_1 and F_2 are equal because they are acting at the same distance from O. Let θ_1 be the angle between F_1 and dr_1.Let θ_2 be the angle between F_2 and dr_2. Then the projection of vectors dr_1 and dr_2 on F_1 and F_2 will be $dr_1 \cos\theta_1$ and $dr_2 \cos\theta_2$. These projections are equal to dr,

Hence $dr_1 \cos\theta_1 = dr_2 \cos\theta_2 = dr$

So $F_1.dr_1 = F_2.dr_2$

(Because $F_1.\, dr_1 = F_1 dr_1 \cos\theta_1$ and $F_2.\, dr_2 = F_2 dr_2 \theta_2$

In the same way we can obtain the same result by considering every path segment taken along path I and path II.

In general $\int_{A(Path1)}^{B} F.\, dr = \int_{A(Path\,II)}^{B} F.\, dr$

Thus the work done by the forces along the two paths is equal $W_{path\ I} = W_{path\ II}$.

In this way, the work done by central force acting on a particle moving from point A and Point B is independent of path. Hence the central force is conservative.

3. *Derive the equation of motion under a Central force.*

Ans: When a body moves under the action of a central force, the force is radial and is always towards a fixed point. The radial acceleration is given by

Radial acceleration = $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ … (1)

In case of a central force, there is no force acting on a particle perpendicular to r, i.e., transverse acceleration is zero.

Hence $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad \Rightarrow r^2\frac{d\theta}{dt} = \text{constant}$

Let $u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

Then $\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt}$

$$= -\frac{1}{u^2}\left(\frac{du}{d\theta}\frac{d\theta}{dt}\right)$$

$$= -\left(r^2\frac{d\theta}{dt}\right)\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

Where $r^2\frac{d\theta}{dt} = h \quad \Rightarrow \frac{d\theta}{dt} = \frac{h}{r^2}$

Further, $\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$

$$= -h\frac{d^2u}{d\theta^2}.\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}.\frac{h}{r^2} = -h^2u^2\frac{d^2u}{d\theta^2} \quad \text{…. (3)}$$

Substituting the value of $\frac{d^2r}{dt^2}$ from eq. (3) in eq. (1),

Radial acceleration = $-h^2u^2\frac{d^2u}{d\theta^2} - r\left(\frac{d\theta}{dt}\right)^2$

$$= -h^2u^2\frac{d^2u}{d\theta^2} - r.\frac{h^2}{r^4}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3$$

Now force acting on the particle = mass x radial acceleration

$\therefore F = -m\left[-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right] = m\left[h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3\right]$

Negative sign indicates that the force is attractive

$\therefore h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = \frac{F}{m} = p$

Where F/m = p is the force per unit mass

$\therefore \frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$ … (4)

This is the differential equation of the orbit of a particle moving under an attractive central force p per unit mass.

4. State and prove Kepler's laws of planetary motion.

Ans: First law: All planets revolve around the Sun in elliptical orbits, having the Sun as one of the foci.

Second law: The radius vector joining the planet to the Sun sweeps equal areas in equal intervals of time. (Or) The areal velocity of radius vector is a constant.

Third law: The square of the time period of revolution of the planet around the sun is directly proportional to the cube of the semi major axis of its elliptical path.

Proof for the First law:consider a planet of mass*m*rotating about the sun of mass M in an orbit of radius *r*. The gravitational force of attraction between them is given by, $F = G\frac{mM}{r^2}$

This force is directed towards the centre of the Sun.
Acceleration of the planet towards the centre of the Sun

$= \frac{Gm}{r^2} = \frac{\mu}{r^2} \quad (\because Gm = \mu)$

The radial acceleration of the planet $= \frac{d^2r}{dt^2} - r\left[\frac{d\theta}{dt}\right]^2$

$$\frac{d^2r}{dt^2} - r\left[\frac{d\theta}{dt}\right]^2 = \frac{\mu}{r^2} \quad \ldots (1)$$

The transverse acceleration of the planet $= \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]$

For the motion under central force the transverse acceleration is zero.

$$\therefore \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] = 0 \quad \Rightarrow r^2\frac{d\theta}{dt} = h \text{ which is a constant}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{h}{r^2} \quad \ldots (2)$$

Let $r = \frac{1}{u} \Rightarrow \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$

$$\left(\because \frac{d\theta}{dt} = \frac{h}{r^2} = hu^2\right)$$

$$\therefore \frac{d^2r}{dt^2} = \frac{d}{dt}\left[-h\frac{du}{d\theta}\right] = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} \quad \ldots (3)$$

Substituting these values in the eq. (1),

$$-h^2u^2\frac{d^2u}{d\theta^2} - r\left[\frac{h}{r^2}\right]^2 = \frac{\mu}{r^2}$$

$$\Rightarrow -h^2u^2\frac{d^2u}{d\theta^2} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} \Rightarrow -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\mu u^2$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \quad \Rightarrow \frac{d^2u}{d\theta^2} + \left[u - \frac{\mu}{h^2}\right] = 0 \quad \ldots (4)$$

Let $x = u - \frac{\mu}{h^2} \Rightarrow \frac{dx}{d\theta} = \frac{du}{d\theta} \Rightarrow \frac{d^2x}{d\theta^2} = \frac{d^2u}{d\theta^2}$

$$\therefore (4) \Rightarrow \frac{d^2x}{d\theta^2} + x = 0$$

The solution of this equation can be written as, $x = a\cos\theta$ where a, θ are constants.

$$\therefore u - \frac{\mu}{h^2} = a\cos\theta \Rightarrow u = \frac{\mu}{h^2} + a\cos\theta \Rightarrow \frac{1}{r} = \frac{\mu}{h^2}\left[1 + \frac{h^2}{\mu}a\cos\theta\right]$$

$$\Rightarrow \frac{1}{r} = \frac{\left(1 + \frac{ah^2}{\mu}\cos\theta\right)}{\left(\frac{h^2}{\mu}\right)}$$

This is similar to the polar equation of the conic

$$\frac{1}{r} = \frac{1 + \varepsilon\cos\theta}{l}$$

Comparing the two, semilatus rectum $l = \frac{h^2}{\mu}$

and eccentricity $\varepsilon = \frac{ah^2}{\mu}$

If $\varepsilon \succ 1$; the conic is hyperbola

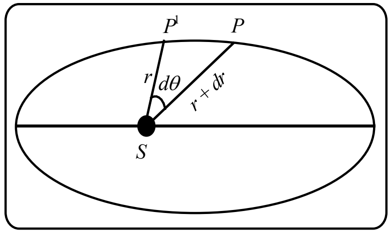
If $\varepsilon = 1$; the conic is parabola

If $\varepsilon \prec 1$; the conic is ellipse.

If the path is either parabola or hyperbola, the planet would pass away from the solar system and will never return to it. Hence all the planetary orbits are elliptical.

This verifies Kepler's first law.

Proof for the second law: let the planet moves from P to P^1 in the time *dt*. Now the area swept by the radius vector is,



$$dA = \frac{1}{2} r^2 d\theta$$

$$\left(\because area\ of\ a\ triangle = \frac{1}{2} \times base \times height \right)$$

Areal velocity $\frac{dA}{dt} = \frac{1}{2} r^2 \left[\frac{d\theta}{dt} \right]$ but $r^2 \left[\frac{d\theta}{dt} \right] = h = constant$

$\therefore$ Areal velocity $\frac{dA}{dt} = \frac{h}{2} = constant$

This verifies Kepler's second law.

Proof for the third law: The time period of revolution of the planet T is given by,

$$T = \frac{area\ swept\ in\ one\ revolution}{areal\ velocity} = \frac{\pi a b}{\frac{h}{2}} = \frac{2 \pi a b}{h}$$

Where a =semi major axis of the ellipse ,
b = semi minor axis of the ellipse.

Squaring on both sides, $T^2 == \frac{4 \pi^2 a^2 b^2}{h^2}$

But eccentricity $\varepsilon = \frac{a h^2}{\mu}$ where $\mu = Gm$ and $h = r^2 \frac{d\theta}{dt}$ are constants $\therefore h^2 = \frac{\varepsilon\mu}{a}$

$$\therefore T^2 = \frac{4 \pi^2 a^2 b^2}{\frac{\varepsilon\mu}{a}} = \frac{4 \pi^2 a^3 b^2}{\varepsilon\mu}$$

$$\Rightarrow T^2 = \left(\frac{4 \pi^2 b^2}{\varepsilon\mu} \right) a^3 \Rightarrow T^2 \propto a^3$$

This verifies Kepler's third law

5. Explain the motion of satellites.

Ans: Satellite: Any relatively smaller body moving round relatively massive body is called as satellite and its closed path is called as orbit.

Ex: Moon is a natural satellite of earth.

Orbital velocity: The velocity of a satellite in its orbit is called orbital velocity.

When a satellite of mass m rotates around earth in a circular orbit of radius *r* , with a velocity *v*,

The centripetal force = $\frac{mv^2}{r}$, gravitational force between earth and satellite = $\frac{G M m}{r^2}$

Where G = gravitational constant, M= mass or the earth

The gravitational force supplies the centripetal force,

$$\frac{G M m}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{G M}{r} \Rightarrow v = \sqrt{\frac{G M}{r}}$$

If g^1 is the acceleration due to gravity at the orbit height we have $g^1 = \frac{GM}{r^2}$

$$\therefore v = \sqrt{g^1 r}$$

This is the expression for the orbital velocity of satellite revolving round the earth at a height h.

If g is acceleration due to gravity on earth,

$$g = \frac{GM}{R^2} \Rightarrow GM = gR^2 \text{ so } g^1 = \frac{g\ R^2}{r^2}$$

$$\therefore v = \sqrt{\frac{g\ R^2}{r^2} r} \Rightarrow v = \sqrt{\frac{g\ R^2}{r}}$$

$$\therefore v = R\sqrt{\frac{g}{(R+h)}}$$

If T is the period of revolution, $T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{g^1 r}} = 2\pi\sqrt{\frac{r}{g^1}}$

In terms of R, h, g we have $T = \frac{2\pi (R+h)^{\frac{3}{2}}}{R\sqrt{g}}$

Angular momentum of the satellite, L = $mvr = m\sqrt{\frac{GM}{r}}$

r= constant

Hence the conservation of angular momentum holds good.

6. Explain the basic idea of Global Positioning System (GPS).

Ans: The Global Positioning System (GPS) is a satellite-based radio navigation system developed and operated by the U.S. Department of Defence. GPS permits land, sea, and

flying users to determine their position, velocity and the time 24 hours a day. It works in all weather conditions, and anywhere in the world. The GPS signals are available to an unlimited number of users simultaneously. The GPS satellites can be used free of charge by anyone.

Each GPS satellite transmits signals to equipment on the ground. GPS receivers passively receive satellite signals, but they do not transmit. GPS receivers require an unobstructed view of the sky, so they are used only outdoors and they might perform less well within forest areas or near tall buildings. GPS operations depend on a very accurate time reference, which is provided by atomic clocks at the U.S. Naval Observatory. Each GPS satellite has atomic clocks on board.

The Global Positioning System (GPS) uses a network of satellites which let people with GPS receivers pinpoint their location anywhere in the world. TomTom is one of the first companies to make GPS technology available in an easy-to-use form for everyone.

Uses pf GPS :

1. GPS is an essential element of the global information infrastructure. The free, open, and dependable nature of GPS has led to the development of hundreds of applications affecting every aspect of modern life.

2. GPS technology is now in everything from cell phones and wristwatches to bulldozers, shipping containers, and ATM's.

3. GPS boosts productivity across a wide range of the economy, to include farming, construction, mining, surveying, package delivery, and logistical supply chain management.

4. Major communications networks, banking systems, financial markets, and power grids depend heavily on GPS for precise time synchronization. Some wireless services cannot operate without it.
5. GPS saves lives by preventing transportation accidents, using search and rescue efforts, and speeding the delivery of emergency services and disaster relief.
6. By the use of GPS the air transportation system in the next generation will enhance flight safety while increasing airspace capacity. GPS also advances scientific aims such as weather forecasting, earthquake monitoring, and environmental protection.
7. GPS is used for national security, and its applications have major role in military operations. Nearly all new military resources from vehicles to weapons come equipped with GPS.

7. Explain the concept of weightlessness.

Ans: Weightlessness is only a sensation, it is not a reality corresponding to an individual who has lost weight. Weightless sensations exist when all contact forces are removedAstronauts orbiting the earth, persons in a freely falling elevator are weightless. They are weightless because there is no external contact force pushing or pulling on them. In these cases, gravity is the only force acting on their body. But it cannot be felt and would not provide any sensation of their weight. But for certain, the orbiting astronauts weigh something. There is a force of gravity acting upon their body. It is the force of gravity that supplies the centripetal force in circular motion. The force of gravity is the only force acting upon their body. The astronauts are in free-fall. Similar to the freely falling elevator, the astronauts and their surroundings are falling towards the Earth under the only influence of gravity. Their tangential velocity allows them to remain in orbital motion while the force of gravity pulls them inward.

Many people believe that orbiting astronauts are weightless because they do not experience a force of gravity. If the absence of gravity is the cause of the weightlessness of orbiting astronauts, it would be in violation of circular motion principles. Hence there must be a force of gravity for there to be an orbit.

The astronauts are weightless because the force of gravity is reduced in space. With less gravity, there would be less weight and thus they would feel less than their normal weight. The force of gravity acting upon an astronaut on the space station is certainly less than on Earth's surface.

8. Explain the Physiological effects of astronauts.

Ans: Astronauts possess a wide range of technical skills related to the space mission and a of behavioural capability that enable them to function in a space orbit environment. For an astronaut the changes in normal physiology due to abnormal environments are called acclimation.The ground-based supporting team works to maintain the Physiologicalpiece of the astronauts by providing support via video teleconferences with family, private psychological conferences and provision of recreational material such as DVDs, books and musical instruments.

The space-flight environment and long-duration space flight can add to tiredness and sleep arrears. So by providing uninterrupted sleep periods, noiseless sleep stations and the sleeping medications can reduce the amount of tiredness experienced by the astronaut.

Acclimation of the cardiovascular system to weightlessness is difficult and not completely understood. Control mechanisms involving the autonomic nervous system, cardiac functions and peripheral vasculature all play a role.

Support is provided after landing to astronauts by health and performance teams. A multinational behaviour and performance committee has made a number of important recommendations to the enhance crew performance. They have implemented family-support programs to help with the many issues faced by astronauts and their families.

SHORT ANSWER QUESTIONS

9. Define a central force . Mention the properties of central forces.

Ans: Definition : A central force is defined as a force which always acts on a particle towards or away from a fixed point and whose magnitude depends only on the distance from the fixed point. This fixed point is known as the center of the force.

Examples: (1) The gravitational force exerted on a particle by another particle which is stationary in an inertial frame of reference s a central force.

(2) The electrostatic force exerted on a charged particle by another stationary, charged particle is a central force.

Properties of Central Forces :

i) The general form of central force is represented by $F = \hat{r} f(r)$, where *f(r)* is a function of distance *r* of the particle from the fixed point and *r* is a unit vector along

the radius vector *r* of the particle with respect to that fixed point.

ii) Central force is a conservation force i.e., the work done by the force in moving a particle from one point to another is independent of the path followed.

iii) Under a central force, the torque acting on the particle is always zero.

iv) Under a central force, the angular momentum of the particle remains conserved.

v) Under a central force, the areal velocity of the particle remains constant.

vi) The central force is attractive when $f(r) < 0$ i.e., negative and repulsive when $f(r) > 0$ i.e., positive.

10. What is a conservative force?

Ans: A force is said to be conservative when the work done by the force in moving a particle from a point A to a point B is independent of the path followed between A and B. Thus the work done by a conservative force between the points A and B is the same for all the paths. The work done depends only on the particle's initial and final position. The work done by a conservative force along a closed path is zero.

11. State Kepler's laws of planetary motion.

Ans: First law: All planets revolve around the Sun in elliptical orbits, having the Sun as one of the foci.

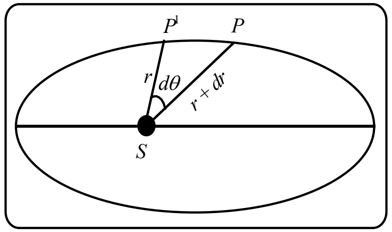
Second law: The radius vector joining the planet to the Sun sweeps equal areas in equal intervals of time. (Or) The areal velocity of radius vector is a constant.

Third law: The square of the time period of revolution of the planet around the sun is directly proportional to the cube of the semi major axis of its elliptical path.

12. State and prove Kepler's second law.

Ans: Second law: The radius vector joining the planet to the Sun sweeps equal areas in equal intervals of time. (Or) The areal velocity of radius vector is a constant.

Proof for the second law: let the planet moves from P to P[1]in the time *dt*. Now the area swept by the radius vector is,



$$dA = \frac{1}{2} r^2 d\theta$$

$$\left(\because area\, of\ a\ triangle = \frac{1}{2} \times base \times height \right)$$

Areal velocity $\frac{dA}{dt} = \frac{1}{2} r^2 \left[\frac{d\theta}{dt} \right]$ but $r^2 \left[\frac{d\theta}{dt} \right] = h = constant$

$\therefore$ Areal velocity $\frac{dA}{dt} = \frac{h}{2} = constant$

This verifies Kepler's second law.

13. State and prove Kepler's third law

Ans: Third law: The square of the time period of revolution of the planet around the sun is directly proportional to the cube of the semi major axis of its elliptical path.

Proof for the third law: The time period of revolution of the planet T is given by,

$$T = \frac{area\ swept\ in\ one\ revolution}{areal\ velocity} = \frac{\pi a b}{\frac{h}{2}} = \frac{2\pi a b}{h}$$

Where a =semi major axis of the ellipse ,
b = semi minor axis of the ellipse.

Squaring on both sides, $T^2 == \frac{4\pi^2 a^2 b^2}{h^2}$

But eccentricity $\varepsilon = \frac{a h^2}{\mu}$ where $\mu = Gm$ and $h = r^2 \frac{d\theta}{dt}$ are constants $\therefore h^2 = \frac{\varepsilon\mu}{a}$

$$\therefore T^2 = \frac{4\pi^2 a^2 b^2}{\frac{\varepsilon\mu}{a}} = \frac{4\pi^2 a^3 b^2}{\varepsilon\mu}$$

$$\Rightarrow T^2 = \left(\frac{4\pi^2 b^2}{\varepsilon\mu} \right) a^3 \Rightarrow T^2 \propto a^3$$

This verifies Kepler's third law

14. Explain the terms satellite and orbital velocity..

Ans: Satellite: Any relatively smaller body moving round relatively massive body is called as satellite and its closed path is called as orbit.

Ex: Moon is a natural satellite of earth.

Orbital velocity: The velocity of a satellite in its orbit is called orbital velocity.

Orbital velocity $v = \sqrt{\frac{GM}{r}}$

At a height h orbital velocity $v = R\sqrt{\frac{g}{(R+h)}}$

If T is the period of revolution, $T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{g^1 r}} = 2\pi\sqrt{\frac{r}{g^1}}$

In terms of R, h, g we have $T = \frac{2\pi(R+h)^{\frac{3}{2}}}{R\sqrt{g}}$

Angular momentum of the satellite, L = $mvr = m\sqrt{\frac{GM}{r}}$

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16. Give the applications of GPS

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SOLVED PROBLEMS

18. The periodic time of Venus is 224.7 days and that of earth is 365.26 days. Find the radio of the major axes of orbits of Venus and earth.

Sol: According to Kepler's III law.

$$T^2 \propto a^2 \qquad \Rightarrow T \propto a^{3/2}$$

Where T is the time period of revolving of planet and 2a the major axes of orbit of ellipse.

$$\therefore \frac{T_{venus}}{T_{earth}} = \left(\frac{a_{venus}}{a_{earth}}\right)^{3/2}$$

$$\Rightarrow \frac{a_{venus}}{a_{earth}} = \left(\frac{224.7}{365.26}\right)^{2/3} = 0.7231$$

19. A satellite travels round a planet at maximum and minimum distances 2×10^7 m and 2×10^7 m respectively. If the speed of satellite at the farthest point be 2×10^3 m sec^{-1}, calculate the speed of satellite at the nearest point.

Sol: Angular momentum L = $mv_1r_1 = mv_2r_2$

$$\Rightarrow \frac{v_2}{v_1} = \frac{r_1}{r_2}$$

$$\Rightarrow v_2 = \frac{v_1r_1}{r_2} = \frac{2\times10^3\times2\times10^7}{10^7} = 4\times10^3\, m/\text{sec}.$$

$\therefore$ The speed of the satellite

$$\therefore \left.\begin{array}{r}\text{The speed of the satellite}\\ \text{at the nearest point}\end{array}\right\} = 4\times10^3\, m/\text{sec}.$$

20. *If the mean distance of Mars from the sun is 1.524 times that of earth, find the period of revolution of Mars about the sun.*

Sol: According to Kelper III law,

$$T^2 \propto a^2 \quad \Rightarrow \quad T \propto a^{3/2}$$

Where T is the time period of revolution of planet and 2a the major axis of ellipse

$$\frac{T_{mars}}{T_{earth}} = \left(\frac{a_{mars}}{a_{earth}}\right)^{3/2} = (1.524)^{3/2} = 1.881$$

As earth revolves round the sun in 1 year and $T_{earth} = 1$ year

$$\therefore T_{mars} = T_{earth} \times 1.1881$$

$$= 1 \times 1.881 = 1.881 \text{ years}$$

21. *The maximum and minimum distances of a comet from the sun are 1.6 × 10[2] and 8 × 10[10]m respectively. If the speed of the comet at the nearest point is 6 × 10[4] m/sec. calculate the speed at the farthest point.*

Sol: In the case of a comet, the angular momentum is conserved. Hence

L = mvr = a constant

$$mv_1r_1 = mv_2r_2$$

$$\Rightarrow v_1r_1 = v_2r_2.$$

Substituting the given values, we get

$$v_1 \times 1.6 \times 10^{12} = \left(6 \times 10^4\right) \times \left(8 \times 10^{10}\right)$$

$$\therefore v_1 = \frac{\left(6 \times 10^4\right) \times \left(8 \times 10^{10}\right)}{1.6 \times 10^{12}} = 3 \times 10^3 \, m/\text{sec}$$

22. *If the earth be one-half of its present distance from the sun, what will be the number of days in a year ?*

Sol: According to Kepler's III law,

$$T^2 \propto a^3$$

Where T is the time period and 2a the major axis of the orbit of the ellipse.

$$\therefore \frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

Here $T_1 = 365$ days; $a_1 = x; a_2 = (1/2)\times$ and $T_2 = ?$

Hence $\left(\frac{365}{T_2}\right)^2 = \left(\frac{x}{\frac{1}{2}x}\right)^3 = 8$

$$\Rightarrow \quad T_2^2 = \frac{365^2}{8}$$

$$\therefore T_2 = 129 \text{ days}$$

23. *The semi-major axes of the orbits of Mercury and Mars are respectively 0.387 and 1.524 in astronomical units. If the period of Mercury is 0.241 year, what is the period of Mars ?*

Sol: $\frac{T_{\text{mercury}}}{T_{\text{mars}}} = \left(\frac{a_{\text{mercury}}}{a_{\text{mars}}}\right)^{3/2} = \left(\frac{0.387}{1.524}\right)^{3/2}$

$$T_{\text{mars}} = T_{\text{mercury}} \times \left(\frac{1.524}{0.387}\right)^{3/2}$$

$$= (0.241 \text{ years}) \times (7.8) = 1.9 \text{ years}$$

24. *The earth moves in an approximately circular orbit with radius 1.5×10^{11} in Mars circles around the sun in its orbit in 687 days. What is the radius of Mars orbit around the sun.*

Sol: According to Kepler's law, $T^2 \propto a^3$ where T is the time period and 2a is the major axis of the orbital of the ellipse. Hence,

$$\left(\frac{T_M}{T_E}\right)^2 = \left(\frac{a_M}{a_E}\right)^3$$

Or $$a_M^3 = (a_E)^3 \left(\frac{T_M}{T_E}\right)^2$$

$\therefore$ $$a_M^3 = (1.5 \times 10^{11})^3 \left(\frac{687}{365}\right)^2$$

Or $$a_M = \left[\frac{(1.5)^3 \times 687 \times 687}{365 \times 365}\right] \times 10^{11}$$

$$\mathbf{= 2.286 \times 10^{11}\,m}$$

25. *Estimate the mass of the sun assuming the orbit of earth round the sun is a circle. The distance between the sun and the earth is 1.49 × 10^{11}m and G = 6.67 × 10^{-11} N-m^2 kg^{-2}*

Sol: We know that the gravitational force of attraction between the sun and the earth is

$$F = \frac{G\,M\,m}{r^2}$$

The centripetal force on the earth = $\frac{mv^2}{r}$

$\therefore$ $$\frac{G\,M\,m}{r^2} = \frac{mv^2}{r}$$

Or $$M = \frac{v^2 r}{G} = \left(\frac{2\pi r}{T}\right)^2 \frac{r}{G}$$

where T is the time period of earth round the sun.

$\therefore$ $$M = \frac{4\pi^2}{T^2 T} r^3$$

$$= \frac{4\pi^2 \times (1.49 \times 10^{11})}{(365 \times 24 \times 60 \times 60)^2 (6.67 \times 10^{-11})}$$

$$\mathbf{= 1.972 \times 10^{30}\,kg}.$$

❖❖❖❖❖

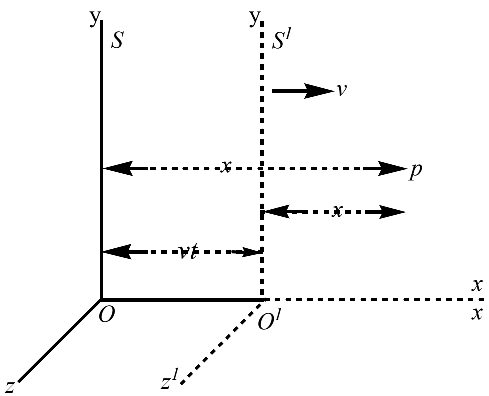
UNIT-III

4. RELATIVISTIC MECHANICS

ESSAY QUESTIONS

1. Derive Galilean transformations.

Ans: Consider a reference frame S with origin O, which is at rest. Let S^1 be another frame with origin O^1. Let S^1 moves relative to S with uniform velocity ' v' in the positive x-direction. At the time t=0 assume that the origins O and O^1 coincides.



Consider an event happening at the point P at a particular time t. Let (x, y, z, t) and (x^1, y^1, z^1, t^1) be the coordinates of P with respect to the frames S and S^1 respectively. In the time y the frame S^1 moves a distance OO^1= vt along the positive x direction. From the figure, OP=x and $O^1P=x^1$.

$$\therefore\ x = x^1 + Vt \quad \Rightarrow x^1 = x - Vt$$

As there is no relative motion along y and z directions, $y=y^1$ and $z=z^1$. According to classical physics, time is supposed to be universal and independent of any reference frame. So $t^1=t$.

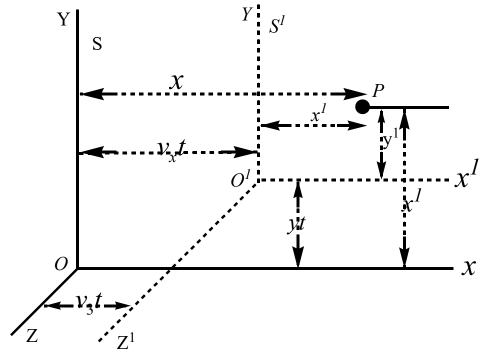
hence $x = x^1 + Vt,\ y = y^1,\ z = z^1,\ t = t^1$ and

$$x^1 = x - Vt,\ y^1 = y,\ z^1 = z,\ t^1 = t$$

The above relations are called Galilean Transformations.

Galilean Velocity Transformations:

Consider the case where S^1 is moving relative to S. Let $\overline{V}$ be the velocity of S^1 relative to S. Then $\overline{V} = v_x\bar{i} + v_y\bar{j} + v_z\bar{k}$.



Where $v_x,\ v_y,\ v_z$ are the components of $\overline{V}$ along X,Y,Z axes. Let (x, y, z, t) and (x^1, y^1, z^1, t^1) be the coordinates of an event with respect to the frames S and S^1 respectively. These coordinates are taken when the frame S^1 is separated from S by the distances $v_x t,\ v_y t,\ v_z t$.

$$x^1 = x - v_x t\ ,\ y^1 = y - v_y t\ ,\ z^1 = z - v_z t$$

Differentiating the above equations w.r.t. t ,

$$\frac{dx^1}{dt} = \frac{dx}{dt} - v_x \quad \Rightarrow u_{x^1} = u_x - v_x,$$

$$\frac{dy^1}{dt} = \frac{dy}{dt} - v_y \quad \Rightarrow u_{y^1} = u_y - v_y,$$

$$\frac{dz^1}{dt} = \frac{dz}{dt} - v_z \quad \Rightarrow u_{z^1} = u_z - v_z$$

Where u_x, u_y, u_z and u_{x^1}, u_{y^1}, u_{z^1} are the velocities in the systems S and S^1 respectively. It can be written as,

$$\overline{u^1} = u_{x^1}\bar{i} + u_{y^1}\bar{j} + u_{z^1}\bar{k} = (u_x - v_x)\bar{i} + (u_y - v_y)\bar{j} + (u_z - v_z)\bar{k}$$

$$\Rightarrow \overline{u^1} = \left(u_x\bar{i} + u_y\bar{j} + u_z\bar{k}\right) - \left(v_x\bar{i} + v_y\bar{j} + v_z\bar{k}\right) \quad \Rightarrow \overline{u^1} = \bar{u} - \bar{V}$$

The above equation represents the Galilean transformation of velocity.

2. *Explain the concept of absolute frame of reference.*

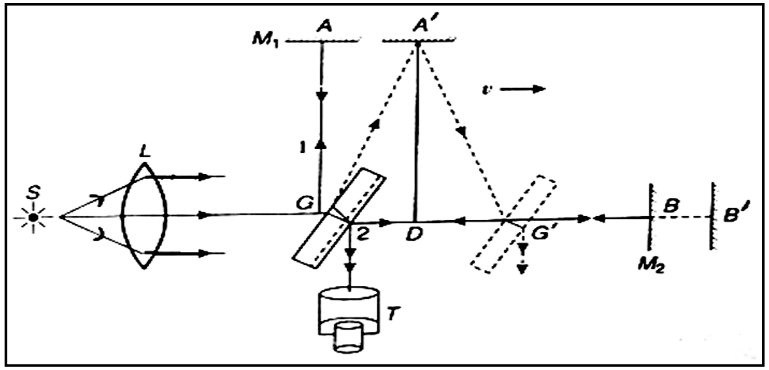
Ans: A frame of reference is a set of coordinates with respect to whom any physical quantity can be determined. An absolute frame of reference is some fixed reference frame that every observer in the universe would agree that, it is at rest at all times. In relativity, no such reference frame exists. It was proved that light travels the same speed for every observer in the universe irrespective of their relative motion. Michelson-Morley conducted an experiment for searching theabsolute frame of reference. But they have a negative result in the experiment. Then Einstein showed that no absolute frame exists.

3. *Explain Michelson – Morley experiment.*

Ans: Search for ether frame *:* With the acceptance of wave theory of light, it was supposed that the space is filled with a substance called as ether. The ether was supposed to be invisible, mass less, perfectly transparent, perfectly non-resistive having high elasticity and low density. Lorentz maintained the stationary ether concept. It was further

supposed that the earth move through it without producing any disturbance. So, if the ether hypothesis is correct then it is possible to determine the absolute velocity of earth with respect to other frame. Michelson and Morley carried out an experiment using Michelson interferometer for this purpose.

Experimental arrangement:Light from a monochromatic extended source S is made parallel by a collimating lens L.It falls on the semi silvered glass plate G inclined at an angle



45^0 to the beam. It is divided into two parts. one is the reflected from the semi silvered surface giving rise to a ray1. It travels towards mirror M_1 and the other is the transmitted ray 2. It travels towards mirror M_2. The two rays fall normally on mirrors M_1 and M_2 respectively and are reflected back along their original path. The reflected rays again meet at the semi silvered surface of glass plate G and enter the telescope where interference pattern is obtained. The optical distances of the mirrors M_1 and M_2 from G are made equal with the help of a compensating plate.

If the apparatus is at rest in ether, the two reflected rays would take equal time to return to the glass plate G. But actually the whole apparatus is moving along with the earth.

Let us suppose that the direction of motion of earth is in the direction of the initial beam. Due to the motion of the earth, the optical paths, traversed by both the rays are not the same. The reflection at the mirrors M_1 and M_2 do not take place at A and B but A^1 and B^1 respectively as shown in the figure. Thus the times taken by the two rays to travel to the mirrors and back to G will be different in this case.

Theory: Let the mirrors M_1 and M_2 are at an equal distance l from the glass plate G. Further 'c' and 'V' be the velocities of light and apparatus or earth respectively. The total path of the ray from G to A_1 and back will be GA^1G^1.

From ΔGA^1D, $\left(GA^1\right)^2 = \left(AA^1\right)^2 + \left(A^1D\right)^2$ … (1)

$$\left(\because GD = AA^1\right)$$

Let 't' be the time taken by the ray to move from G to A^1, then from the equation (1) we have

$$(ct)^2 = (Vt)^2 + (l)^2 \Rightarrow t^2\left(c^2 - V^2\right) = l^2 \Rightarrow t = \frac{l}{\sqrt{\left(c^2 - V^2\right)}}$$

If t_1 be the time taken by the ray to travel the whole path GA^1G, then

$$t_1 = 2t = \frac{2l}{\sqrt{\left(c^2 - V^2\right)}} = \frac{2l}{c\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}} = \frac{2l}{c}\left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$

$$= \frac{2l}{c}\left(1 + \frac{V^2}{2c^2}\right) \quad \text{… (2)}$$

In case of the ray 2 which is moving towards mirror M_2, the relative velocity is (C-V) when it is moving from G to B.

from B to G it will be (C+V). If t_2 be the total time taken by the ray 2for its travel, then

$$t_2 = \frac{l}{c - V} + \frac{l}{c + V} \qquad \left(\because GB = G^1B = l\right)$$

$$t_2 = \frac{l(c+V) + l(c-V)}{\left(c^2 - V^2\right)} = \frac{2lc}{\left(c^2 - V^2\right)} = \frac{2lc}{c^2\left(1 - \frac{V^2}{c^2}\right)}$$

$$= \frac{2l}{c}\left(1 - \frac{V^2}{c^2}\right)^{-1} = \frac{2l}{c}\left(1 + \frac{V^2}{c^2}\right) \quad \text{… (3)}$$

Thus the difference in times of travel of longitudinal and transverse path is

$$\Delta t = t_2 - t_1 = \frac{2l}{c}\left(1 + \frac{V^2}{c^2}\right) - \frac{2l}{c}\left(1 + \frac{1}{2}\frac{V^2}{c^2}\right)$$

$$= \frac{2l}{c}\frac{V^2}{2c^2} = \frac{lV^2}{c^3} \quad \text{… (4)}$$

$\therefore$ Optical path difference between two rays is given by:

Optical path difference = velocity $\times \Delta t = c \times \Delta t$

$$= c\left(\frac{lV^3}{c^3}\right) = \frac{lV^2}{c^2}$$

If λ is the wavelength of light used, then Path difference in terms of wavelength = $\frac{lV^2}{c^2\lambda}$

Michelson and Morley performed the experiment in two steps i.e. the setting shown in Fig. and secondly by turning the apparatus through 90^0. When the apparatus was turned through 90^0, the positions of two mirrors are changed. Now the path difference is in opposite directions i.e. the path difference is $\left(-\frac{lV^2}{c^2\lambda}\right)$. The resultant path difference now becomes $\frac{lV^2}{c^2\lambda}-\left(-\frac{lV^2}{c^2\lambda}\right)=\frac{2lV^2}{c^2\lambda}$. Weknow that a change in optical path difference by λ corresponds to a shift of one fringe and hence the path difference $\left(\frac{2lV^2}{\lambda}\right)$ corresponds to a fringe shift of $\left(\frac{2lV^2}{\lambda}\right)$

In Michelson and Morley experiment: $l = 1.0 \times 10^3$ cm, $\lambda = 5.0 \times 10^{-5}$ cm, $V = 3 \times 10^6$ cm/sec, and $c = 3 \times 10^{10}$ cm/sec.

$$\therefore \text{Change in fringe shift } n=\frac{2lV^2}{c^2\lambda}$$

$$\Rightarrow n=\frac{2\left(1\times10^3\right)\left(3\times10^6\right)^2}{\left(3\times10^{10}\right)^2\left(5.0\times10^{-5}\right)}=0.4\ fringe$$

Thus a shift of less than half a fringe was only expected. Michelson and Morley could observe a shift of about 0.01 of fringe. This shift is within the limits of the error of observations. They repeated the experiment at different points on the earth's surface and at different seasons of the year but they could not detect any measurable shift. So it was a null or negative result.

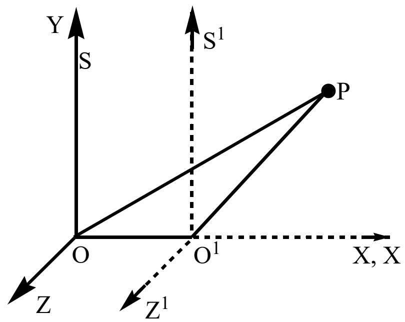
Significance: this experiment proved that the ether concept is not true and the velocity of light is invariant.

4. *Give postulates of special theory of relativity. Derive Lorentz transformations.*

Ans: 1. All physical laws are the same in all inertial frames of reference which are moving with constant velocity relative to each other.

2. The speed of light in vacuum is the same in every inertial frame.

Lorentz transformations of space and time: Consider two inertial systems S and S^1 as shown in the figure. The system S^1 is moving with a velocity v with respect to S in the positive X- direction. Let the axis of the two coordinate systems coincide at $t=t^1=0$. Let a pulse of light be generated at a time t = o at the origin which grows in the space. Now consider the situation when the pulse reaches at point P. Let (x, y, z, t) and (x^1, y^1, z^1, t^1) be the coordinates of P measured by the observers O and O^1 in frames S and S^1 respectively. When the pulse is observed from S,



$$\text{Velocity of light} = \frac{\text{distance}}{time} \Rightarrow c=\frac{\sqrt{x^2+y^2+z^2}}{t}$$

$$\Rightarrow ct=\sqrt{x^2+y^2+z^2} \Rightarrow x^2+y^2+z^2=c^2t^2$$

$$\Rightarrow x^2+y^2+z^2-c^2t^2=0 \qquad \dots (1)$$

When the pulse is observed from S^1 we have,

$$\left(x^1\right)^2+\left(y^1\right)^2+\left(z^1\right)^2-c^2\left(t^1\right)^2=0 \qquad \dots (2)$$

$$\left(\because c \text{ is a constant}\right) \text{ and } \quad y = y^1, z = z^1 \qquad \ldots (3)$$

From equations (1) and (2), we have,

$$x^2 + y^2 + z^2 - c^2t^2 = \left(x^1\right)^2 + \left(y^1\right)^2 + \left(z^1\right)^2 - c^2\left(t^1\right)^2$$

$$\Rightarrow x^2 - c^2t^2 = \left(x^1\right)^2 . - c^2\left(t^1\right)^2 \qquad \ldots (4)$$

Let the transformation between x and x^1 can be represented by the relationship, $x^1 = K\left(x - Vt\right)$ … (5)

where k being independent of x and t.

If we suppose that the system S is moving relative to S^1 with a velocity (–V) along positive X direction then,

$$x = k\left(x^1 + V t^1\right) \qquad \ldots (6)$$

$$\Rightarrow \frac{x}{k} = \left(x^1 + V t^1\right)$$

Substituting the value of x^1 from eq. (5) in eq. (6), we get

$$\Rightarrow \frac{x}{k} = k\left(x - Vt\right) + V t^1 \Rightarrow \frac{x}{k} - k\left(x - V t\right) = V t^1$$

$$\Rightarrow \frac{x}{k} - k x + k V t = V t^1$$

$$\Rightarrow t^1 = \frac{x}{kV} - \frac{k x}{V} + k t \Rightarrow t^1 = \frac{k x}{V}\left(\frac{1}{k^2} - 1\right) + k t$$

$$\Rightarrow t^1 = k\left[\frac{x}{V}\left(\frac{1}{k^2} - 1\right) + t\right] \qquad \ldots (7)$$

Substituting the value of x^1 from eq. (5) and t^1 from eq. (7) in eq. (4), we get

$$(4) \Rightarrow x^2 - c^2 t^2 = \left(x^1\right)^2 - c^2\left(t^1\right)^2$$

$$\Rightarrow x^2 - c^2t^2 = k^2\left(x - V t\right)^2 - c^2 k^2\left[\frac{x}{V}\left(\frac{1}{k^2} - 1\right) + t\right]^2$$

$$\Rightarrow x^2 - c^2t^2 = k^2\left(x^2 + V^2t^2 - 2\, xV t\right) - c^2 k^2$$

$$\left[\frac{x^2}{V^2}\left(\frac{1}{k^2} - 1\right)^2 + t^2 + 2t\frac{x}{V}\left(\frac{1}{k^2} - 1\right)\right]$$

Equating the coefficients of t^2, $-c^2 = k^2V^2 - c^2 k^2$

$$\Rightarrow -c^2 = k^2c^2\left(\frac{V^2}{c^2} - 1\right)$$

$$\Rightarrow k^2 = \frac{1}{\left(1 - \frac{V^2}{c^2}\right)} \quad \Rightarrow k = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Substituting the value of K in eq. (5), we have the Lorentz transformation for space i.e.

$$(5) \Rightarrow x^1 = K\left(x - Vt\right) \quad \Rightarrow x^1 = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad \ldots (9)$$

$$(7) \Rightarrow t^1 = k\left[t + \frac{x}{V}\left(\frac{1}{k^2} - 1\right)\right]$$

$$\Rightarrow t^1 = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[t + \frac{x}{V}\left(1 - \frac{V^2}{c^2} - 1\right)\right] \Rightarrow t^1 = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[t - \frac{x V}{c^2}\right]$$

So, the Lorentz transformations are,

$$x^1 = \frac{x - V\,t}{\sqrt{1-\frac{V^2}{c^2}}},\ y=y^1,\ z=z^1,\ t^1 = \frac{\left(t - \frac{x\,V}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

Lorentz inverse transformation: If we assume that the system S is moving with velocity (-V) relative to S^1 along positive direction of X, then the Lorentz transformation equations can be expressed as

$$x = \frac{x^1 + V\,t^1}{\sqrt{1-\frac{V^2}{c^2}}},\ y^1=y,\ z^1=z,\ t = \frac{\left(t^1 + \frac{x^1\,V}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

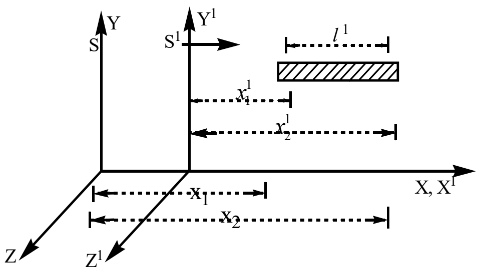
These are known as inverse Lorentz transformation equations.

5. Explain the concept of length contraction.

Ans: Consider two coordinate systems S and S^1. Let S^1 be moving with velocity v relative to S, along the positive X-direction. Let a rod be placed in S^1 along X-axis. Let x_1, x_2 be the X-coordinates at the ends of the rod w.r.t. S-frame.Then the length of the rod is given by,

$l = x_2 - x_1$ ... (1)

Let x_1^1, x_2^1 be the X-coordinates of the ends of the rod w.r.t. S^1-frame. Then its length l^1 in the system S^1 is given by,



$$l^1 = x_2^1 - x_1^1 \quad \ldots (2)$$

The situation is shown in fig. According to Lorentz transformation equation, we get,

$$x_2^1 = \frac{x_2 - V\,t}{\sqrt{1-\frac{V^2}{c^2}}} \quad \text{and} \quad x_1^1 = \frac{x_1 - V\,t}{\sqrt{1-\frac{V^2}{c^2}}}$$

Substituting these values in eq. (1), we have

$$l^1 = \frac{(x_2 - Vt)}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}} - \frac{(x_1 - Vt)}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}} = \frac{(x_2 - x_1)}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}} = \frac{l}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}}$$

$$\Rightarrow \quad l = l^1\sqrt{\left(1-\frac{V^2}{c^2}\right)}$$

Thus the length of the rod moving with velocity V relative to the observer is contracted by a factor $\sqrt{\left(1-\frac{V^2}{c^2}\right)}$ in the direction of motion. This is known as Lorentz-Fitzerald length contraction.

6. Explain the concept of time dilation.

Ans: Dilation means to lengthen an interval of time. Consider two systems S and S^1. Let S^1 be moving with a velocity V with respect to S in the positive direction of X-axis. Suppose a clock is placed in the system S at position x and gives signals of intervals Δt.

i.e., $\Delta t = t_2 - t_1$ ... (1)

If this time interval is observed by an observer in system S^1 then interval Δt^1 recorded by him given by $\Delta t^1 = t_2^1 - t_1^1 \ldots (2)$

From Lorentz transformation, we have $t_1^1 = \dfrac{t_1 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}}$ … (3)

and $t_2^1 = \dfrac{t_2 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}}$ … (4)

Substituting these values of t_2^1 and t_1^1 from equations (3) and (4) in equation (2), we get,

$$\Delta t^1 = \frac{t_2 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}} - \frac{t_1 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}}$$

$$= \frac{t_2 - t_1}{\sqrt{1-\dfrac{V^2}{c^2}}} = \frac{\Delta t}{\sqrt{1-\dfrac{V^2}{c^2}}}$$

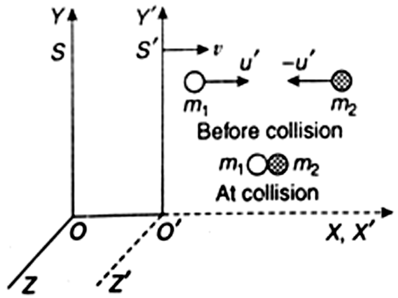
$$\therefore \Delta t^1 = \frac{\Delta t}{\sqrt{1-\dfrac{V^2}{c^2}}} \quad \text{… (5)}$$

This equation shows that $\Delta t^1 < \Delta t$ i.e. the time interval in system S is greater than the time interval in system S^1.

7. Explain variation of mass with velocity.

Ans: Consider two coordinate systems S and S^1. Let S^1 be moving with velocity v relative to S, along the positive X-direction, consider the collision of two bodies on the system S^1 andview it from S. let the two bodies of masses

$m_1$and m_2 move with velocities u^1 and $(-u^1)$ parallel to X-axis in the system S^1.letr the two bodies collide and merge in to one. As the two bodies were moving with the same velocity in opposite directions, they are at rest after collision in S^1. Let u_1 and u_2 be the initial velocities relative to the system S and v be their common velocity after collision. Then according to the addition of velocities,



$$u_1 = \frac{u^1 + v}{1 + \dfrac{u^1 v}{c^2}} \text{ and } u_2 = \frac{-u^1 + v}{1 - \dfrac{u^1 v}{c^2}} \quad \text{… (1)}$$

Applying the law of conservation of momentum,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)\, v \quad \text{… (2)}$$

Substituting the values of u_1 and u_2 in the equation (2),

$$m_1 \frac{u^1 + v}{1 + \dfrac{u^1 v}{c^2}} + m_2 \frac{-u^1 + v}{1 - \dfrac{u^1 v}{c^2}} = (m_1 + m_2)\, v$$

$$\Rightarrow m_1 \left[\frac{u^1 + v}{1 + \dfrac{u^1 v}{c^2}} - v\right] = m_2 \left[v - \frac{-u^1 + v}{1 - \dfrac{u^1 v}{c^2}}\right]$$

$$\Rightarrow m_1 \left[\frac{u^1 + v - v - \dfrac{u^1 v^2}{c^2}}{1 + \dfrac{u^1 v}{c^2}}\right] = m_2 \left[\frac{v - \dfrac{u^1 v^2}{c^2} + u^1 - v}{1 - \dfrac{u^1 v}{c^2}}\right]$$

$$\Rightarrow m_1\left[\frac{u^1\left(1-\frac{v^2}{c^2}\right)}{1+\frac{u^1v}{c^2}}\right]=m_2\left[\frac{u^1\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u^1v}{c^2}}\right]$$

$$\Rightarrow \frac{m_1}{m_2}=\left[\frac{1+\frac{u^1v}{c^2}}{1-\frac{u^1v}{c^2}}\right] \qquad \ldots(3)$$

We have $(1)\Rightarrow u_1=\dfrac{u^1+v}{1+\frac{u^1v}{c^2}} \quad \Rightarrow u_1^{\,2}=\dfrac{\left(u^1+v\right)^2}{\left(1+\frac{u^1v}{c^2}\right)^2}$

$$\Rightarrow \frac{u_1^{\,2}}{c^2}=\frac{1}{c^2}\frac{\left(u^1+v\right)^2}{\left(1+\frac{u^1v}{c^2}\right)^2}$$

$$\Rightarrow 1-\frac{u_1^{\,2}}{c^2}=1-\frac{1}{c^2}\frac{\left(u^1+v\right)^2}{\left(1+\frac{u^1v}{c^2}\right)^2}$$

$$=\frac{\left(1+\frac{u^1v}{c^2}\right)^2-\left(\frac{u^1+v}{c}\right)^2}{\left(1+\frac{u^1v}{c^2}\right)^2}=\frac{1+\left(u^1\right)^2v^2}{c^4}$$

$$+\frac{2u^1v}{c^2}-\frac{\left(u^1\right)^2}{c^2}-\frac{v^2}{c^2}-\frac{2u^1v}{c^2}$$

$$=\frac{1+\frac{\left(u^1\right)^2v^2}{c^4}-\frac{\left(u^1\right)^2}{c^2}-\frac{v^2}{c^2}}{\left(1+\frac{u^1v}{c^2}\right)^2}=\frac{\left(1-\frac{\left(u^1\right)^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u^1v}{c^2}\right)^2}$$

$$\therefore\ 1-\frac{u_1^{\,2}}{c^2}=\frac{\left(1-\frac{\left(u^1\right)^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u^1v}{c^2}\right)^2}$$

$$1+\frac{u^1v}{c^2}=\left(\frac{\left(1-\frac{\left(u^1\right)^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_1^{\,2}}{c^2}}\right)^{\frac{1}{2}} \quad \text{Similarly}$$

$$1-\frac{u^1v}{c^2}=\left(\frac{\left(1-\frac{\left(u^1\right)^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_2^{\,2}}{c^2}}\right)^{\frac{1}{2}}$$

Substituting these values in the equation (3),

$$(3)\Rightarrow\frac{m_1}{m_2}=\left[\frac{1+\frac{u^1v}{c^2}}{1-\frac{u^1v}{c^2}}\right]$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{\left(\left(1-\frac{(u^1)^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)\Big/ 1-\frac{u_1^2}{c^2}\right)^{\frac{1}{2}}}{\left(\left(1-\frac{(u^1)^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)\Big/ 1-\frac{u_2^2}{c^2}\right)^{\frac{1}{2}}} \Rightarrow \frac{m_1}{m_2} = \frac{\left(1-\frac{u_2^2}{c^2}\right)^{\frac{1}{2}}}{\left(1-\frac{u_1^2}{c^2}\right)^{\frac{1}{2}}}$$

If the mass m_2 is at rest in the system S before collision, then $u_2 = 0$

$$\therefore \frac{m_1}{m_2} = \frac{1}{\sqrt{1-\frac{u_1^2}{c^2}}}$$ If $m_1 = m$ and $m_2 = m_0$, then

$$\therefore m = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

This gives the relativistic variation of mass with velocity.

8. Explain Einstein's mass energy relation

Ans: According to classical mechanics, the energy is defined in terms of work (Force x distance) and the force is the rate of change of momentum, hence

$$F = \frac{d}{dt}(mV) \quad \ldots (1)$$

According to theory of relativity, the mass as well as velocity are variable, thus,

$$F = m\frac{dV}{dt} + V\frac{dm}{dt} \quad \ldots (2)$$

When a particle is displaced through a distance dx by the application of a force F, then the increase in kinetic energy, dK is given by, $dK = Fdx$ … (3)

Substituting the value of F from eq. (2) in eq. (3), we get

$$dK = m\frac{dV}{dt}dx + V\frac{dm}{dt}dx$$

$$\Rightarrow dK = mVdV + V^2 dm \quad \ldots (4)$$

$$\left(\therefore \frac{dx}{dt} = V\right)$$

The variation of mass with velocity is given by,

$$m = \frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}}$$

Squaring both sides, we have $m^2 = \frac{m_0^2 c^2}{c^2 - V^2}$

$$\Rightarrow m^2c^2 - m^2V^2 = m_0^2 c^2$$

Differentiating this eq., $c^2 2m\, dm - V^2 2m\, dm - m^2 2V\, dV = 0$

$$\Rightarrow c^2 dm - V^2\, dm - mV\, dV = 0$$

$$\Rightarrow c^2\, dm = V^2\, dm + mVdV \quad \ldots (5)$$

Comparing equations (4) and (5), we have

$dK = c^2\, dm$ ... (6)

Now consider that the body is at rest initially and by the application of force it acquires a velocity V. The mass of the body increases from m_0 to m. The total kinetic energy acquired the body is given by,

$$\int dK = \int_{m_0}^{m} c^2 dm \quad \Rightarrow \quad K = c^2(m - m_0) \quad \ldots (7)$$

This is the increases in kinetic energy due to increase in mass. We know that total energy of a moving particle is the sum of its kinetic energy of motion and the energy at rest. Hence

Total energy $E = K + m_0 c^2 \Rightarrow E = c^2 (m - m_0) + m_0 c^2$

$$\Rightarrow E = mc^2 \quad \dots (8)$$

9. *Explain the concept relativistic of addition of velocities.*

Ans: Consider two systems S and S^1. Let S^1 be moving with a velocity V with respect to S in the positive direction of X-axis. Let a body moves a distance dx in the time dt in the system S and moves a distance dx^1 in time dt^1 in the system s^1. Then $u = \frac{dx}{dt}$ and $u^1 = \frac{dx^1}{dt}$

According to Lorentz transformation, $x = k(x^1 + Vt^1)$,

$$t = k\left(t^1 + \frac{x^1 V}{c^2}\right)$$

Differing the above equations, we have,

$$dx = k(dx^1 + V\,dt^1) \text{ and } dt = k\left(dt^1 + \frac{dx^1 V}{c^2}\right)$$

Dividing the two equations $\frac{dx}{dt} = \frac{k(dx^1 + V\,dt^1)}{k\left(dt^1 + \frac{dx^1 V}{c^2}\right)} = \frac{\frac{dx^1}{dt^1} + V}{1 + \left(\frac{dx^1}{dt^1}\right)\frac{V}{c^2}}$

$$\Rightarrow u = \frac{u^1 + V}{1 + \frac{u^1 V}{c^2}}$$

The above equation represents the relativistic addition of velocities.

SHORT ANSWER QUESTIONS

10. *Give Galilean transformations.*

Ans: hence $x = x^1 + Vt,\ y = y^1,\ z = z^1,\ t = t^1$ and

$$x^1 = x - Vt,\ y^1 = y,\ z^1 = z,\ t^1 = t$$

The above relations are called Galilean Transformations.

Galilean Velocity Transformations:

$$\overline{u^1} = (u_x\bar{i} + u_y\bar{j} + u_z\bar{k}) - (v_x\bar{i} + v_y\bar{j} + v_z\bar{k}) \quad \Rightarrow \overline{u^1} = \bar{u} - \bar{V}$$

The above equation represents the Galilean transformation of velocity.

11. *Explain the concept of absolute frame of reference.*

Ans: A frame of reference is a set of coordinates with respect to whom any physical quantity can be determined. An absolute frame of reference is some fixed reference frame that every observer in the universe would agree that, it is at rest at all times. In relativity, no such reference frame exists. It was proved that light travels the same speed for every observer in the universe irrespective of their relative motion. Michelson-Morley conducted an experiment for searching theabsolute frame of reference. But they have a negative result in the experiment. Then Einstein showed that no absolute frame exists.

12. Give postulates of special theory of relativity. Mansion Lorentz transformations.

Ans: 1. All physical laws are the same in all inertial frames of reference which are moving with constant velocity relative to each other.

2. The speed of light in vacuum is the same in every inertial frame.

Lorentz transformations of space and time:

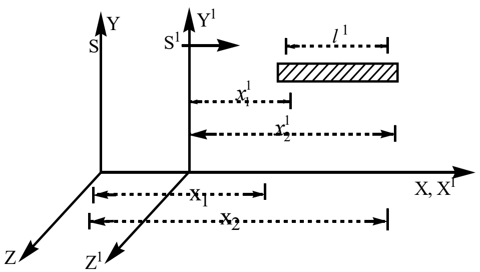
$$x^1 = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}},\ y=y^1,\ z=z^1,\ t^1 = \frac{\left(t - \frac{xV}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

Lorentz inverse transformation:

$$x = \frac{x^1 + Vt^1}{\sqrt{1-\frac{V^2}{c^2}}},\ y^1=y,\ z^1=z,\ t = \frac{\left(t^1 + \frac{x^1 V}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

13. Explain the concept of length contraction.

Ans: Consider two coordinate systems S and S^1. Let S^1 be moving with velocity v relative to S, along the positive X-direction. Let a rod be placed in S^1 along X-axis. Let x_1, x_2 be the X-coordinates at the ends of the rod w.r.t. S-frame. Then the length of the rod is given by,



$$l = x_2 - x_1 \qquad \ldots (1)$$

Let x_1^1, x_2^1 be the X-coordinates of the ends of the rod w.r.t. S^1-frame. Then its length l^1 in the system S^1 is given by,

$$l^1 = x_2^1 - x_1^1 \qquad \ldots (2)$$

The situation is shown in fig. According to Lorentz transformation equation, we get,

$$x_2^1 = \frac{x_2 - Vt}{\sqrt{1-\frac{V^2}{c^2}}} \quad \text{and} \quad x_1^1 = \frac{x_1 - Vt}{\sqrt{1-\frac{V^2}{c^2}}}$$

Substituting these values in eq. (1), we have

$$l^1 = \frac{(x_2 - Vt)}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}} - \frac{(x_1 - Vt)}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}} = \frac{(x_2 - x_1)}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}} = \frac{l}{\sqrt{\left(1-\frac{V^2}{c^2}\right)}}$$

$$\Rightarrow l = l^1\sqrt{\left(1-\frac{V^2}{c^2}\right)}$$

Thus the length of the rod moving with velocity V relative to the observer is contracted by a factor $\sqrt{\left(1-\frac{V^2}{c^2}\right)}$ in the direction of motion. This is known as Lorentz-Fitzerald length contraction.

14. Explain the concept of time dilation.

Ans: Dilation means to lengthen an interval of time. Consider two systems S and S^1. Let S^1 be moving with a velocity V with respect to S in the positive direction of X-axis. Suppose a clock is placed in the system S at position x and gives signals of intervals Δt.

i.e., $\Delta t = t_2 - t_1$ ... (1)

If this time interval is observed by an observer in system S^1 then interval Δt^1 recorded by him given by $\Delta t^1 = t_2^1 - t_1^1 \ldots (2)$

From Lorentz transformation, we have $t_1^1 = \dfrac{t_1 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}} \ldots$ (3)

and $t_2^1 = \dfrac{t_2 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}}$ ... (4)

Substituting these values of t_2^1 and t_1^1 from equations (3) and (4) in equation (2), we get, $\Delta t^1 = \dfrac{t_2 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}} - \dfrac{t_1 - \left(\dfrac{Vx}{c^2}\right)}{\sqrt{1-\dfrac{V^2}{c^2}}}$

$$= \frac{t_2 - t_1}{\sqrt{1-\dfrac{V^2}{c^2}}} = \frac{\Delta t}{\sqrt{1-\dfrac{V^2}{c^2}}}$$

$$\therefore \ \Delta t^1 = \frac{\Delta t}{\sqrt{1-\dfrac{V^2}{c^2}}} \quad \ldots (5)$$

This equation shows that $\Delta t^1 < \Delta t$ i.e. the time interval in system S is greater than the time interval in system S^1.

15. Explain the concept relativistic of addition of velocities.

Ans: Consider two systems S and S^1. Let S^1 be moving with a velocity V with respect to S in the positive direction of X-axis. Let a body moves a distance dx in the time dt in the system S and moves a distance dx^1 in time dt^1 in the system s^1. Then $u = \dfrac{dx}{dt}$ and $u^1 = \dfrac{dx^1}{dt}$

According to Lorentz transformation, $x = k\left(x^1 + Vt^1\right)$,

$$t = k\left(t^1 + \frac{x^1 V}{c^2}\right)$$

Differing the above equations, we have,

$$dx = k\left(dx^1 + V\,dt^1\right) \text{ and } dt = k\left(dt^1 + \frac{dx^1 V}{c^2}\right)$$

Dividing the two equations $\dfrac{dx}{dt} = \dfrac{k\left(dx^1 + V\,dt^1\right)}{k\left(dt^1 + \dfrac{dx^1 V}{c^2}\right)} = \dfrac{\dfrac{dx^1}{dt^1} + V}{1 + \left(\dfrac{dx^1}{dt^1}\right)\dfrac{V}{c^2}}$

$$\Rightarrow u = \frac{u^1 + V}{1 + \dfrac{u^1 V}{c^2}}$$

The above equation represents the relativistic addition of velocities.

SOLVED PROBLEMS

16. *A particle of mass m_0 is moving with a velocity 0.9c. Calculate its relativistic mass and its kinetic energy.*

Sol: We have relativistic mass $m=\frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{(0.9c)^2}{c^2}}}$

$$=\frac{m_0}{\sqrt{1-0.81}}=(2.294)m_0$$

$$KE=mc^2-m_0c^2=2.29m_0c^2-m_0c^2=1.294m_0c^2$$

17. *If the total energy of a particle is exactly thrice its rest energy, what is the velocity of the particle?*

Sol: Given $E=mc^2=2m_0c^2 \Rightarrow m=2m_0$

We have relativistic mass $m=\frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}} \Rightarrow 2m_0=\frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}}$

$$\Rightarrow 2=\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\Rightarrow 1-\frac{V^2}{c^2}=\frac{1}{4}$$

$$\Rightarrow \frac{V^2}{c^2}=1-\frac{1}{4}=\frac{3}{4}\Rightarrow V^2=\left(\frac{3}{4}\right)c^2 \Rightarrow V=\frac{\sqrt{3}c}{2}=0.866c$$

18. *What will be the fringe shift in Michelson-Morley experiment if the effective length of each path is 6 m and light wavelength of 6000A^0 is used? Earth's velocity is 3 × 10^4 ms^{-1} and c = 3 × 10^8 ms^{-1}*

Sol: fringe shift n = $\frac{2lV^2}{c^2\lambda}$

$$\Rightarrow n=\frac{2(6)(3\times10^4)^2}{(3\times10^8)^2(6000\times10^{-10})}=0.2\ fringe$$

19. *At what speed the mass of an object will be double of its value at rest?*

Sol: We have relativistic mass $m=\frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}}$ $given, m=2m_0$

$$\therefore 2m_0=\frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}}\Rightarrow \sqrt{1-\frac{V^2}{c^2}}=\frac{1}{2}$$

$$\Rightarrow 1-\frac{V^2}{c^2}=\frac{1}{4}\Rightarrow\frac{V^2}{c^2}=\frac{3}{4}\Rightarrow V=\left(\frac{\sqrt{3}}{2}\right)c=2.596\times10^8\,ms^{-1}$$

20. *A clock showing correct time when at rest, loses 1 hour in a day when it is moving. What is its speed?*

Sol: We have, $\therefore \Delta t^1=\frac{\Delta t}{\sqrt{1-\frac{V^2}{c^2}}}\Rightarrow 24=\frac{23}{\sqrt{1-\frac{V^2}{c^2}}}$

$$\Rightarrow \sqrt{1-\frac{V^2}{c^2}} = \frac{23}{24} = 0.9583$$

$$\Rightarrow 1-\frac{V^2}{c^2} = 0.9183 \Rightarrow \frac{V^2}{c^2} = 1-0.9183 = 0.0817$$

$$\Rightarrow V = \left(\sqrt{0.0817}\right)c = (0.286)c$$

21. Find the mass of an electron moving with a velocity of 1 × 10^10 cm/sec. The rest mass of the electron is 9.1 × 10^-31 kg.

Sol: We have relativistic mass $m = \frac{m_0}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{9.1\times10^{-31}}{\sqrt{1-\frac{(1\times10^{-8})^2}{(3\times10^{-8})^2}}}$

$$= \left(9.652\times10^{-31}\right) kg$$

22. A rod of 1 m length is moving with a velocity of 0.6 x 10^8 m/s with respect to a stationary observer. Find the length of the rod (in m) along its direction of motion as seen by the observer.

Sol: we have $l = l^1\sqrt{\left(1-\frac{V^2}{c^2}\right)} = 1\sqrt{1-\frac{(0.6\times10^8)^2}{(3\times10^8)^2}} =$

$$\sqrt{1-\frac{0.36\times10^{16}}{9\times10^{16}}} = \sqrt{1-\frac{4}{100}} = \sqrt{\frac{96}{100}} = 0.9798\,m$$

23. Calculate the expected fringe shifts in the Michelson-Morley experiment, if the distance of each plate is 2 m and the wavelength of monochromatic radiation is (a) 6000A^0 and (b) 4000A^0

Sol: fringe shift n = $\frac{2lV^2}{c^2\lambda} =$

$$\Rightarrow n = \frac{2(2)\left(3\times10^4\right)^2}{\left(3\times10^8\right)^2\left(6000\times10^{-10}\right)} = 0.067\ fringe$$

fringe shift n = $\frac{2lV^2}{c^2\lambda} = \Rightarrow n = \frac{2(2)\left(3\times10^4\right)^2}{\left(3\times10^8\right)^2\left(4000\times10^{-10}\right)} = 0.1\ fringe$

24. If a rod travels with a speed v=0.6 c along its length, calculate the percentage of contraction.

Sol: We have $l = l^1\sqrt{\left(1-\frac{V^2}{c^2}\right)}$

$$\Rightarrow l = l^1\sqrt{1-\frac{(0.6\,c)^2}{c^2}} = l^1\sqrt{1-0.36} = l^1\sqrt{0.64} = (0.8)\,l^1$$

amount of contraction $= l^1 - l = l^1 - (0.8)\,l^1 = 0.2\,l^1$

$$\%\,contraction,\ \frac{l^1 - l}{l^1}\times100 = \frac{0.2\,l^1}{l^1}\times100 = 20\%$$

25. Calculate the velocity of the rod when its length will appear 90% of its proper length.

Sol: We have $l = l^1\sqrt{\left(1-\frac{V^2}{c^2}\right)} \Rightarrow 90 = 100\sqrt{\left(1-\frac{V^2}{c^2}\right)}$

$$\Rightarrow \frac{90}{100} = \sqrt{\left(1-\frac{V^2}{c^2}\right)}$$

$$1-\frac{V^2}{c^2} = \left(\frac{9}{10}\right)^2 = \frac{81}{100} \quad \Rightarrow \frac{V^2}{c^2} = 1-\frac{81}{100} = \frac{19}{100}$$

$$\Rightarrow V^2 = \left(\frac{19}{100}\right)c^2 \quad \Rightarrow V = 1.307\times10^8\, ms^{-1}$$

26. A rocket ship is 100 m long on the ground. When it is in flight, its length is 99 m to an observer on the ground. What is its speed?

Sol: We have $l = l^1\sqrt{\left(1-\frac{V^2}{c^2}\right)} \Rightarrow 99 = 100\sqrt{\left(1-\frac{V^2}{c^2}\right)}$

$$\Rightarrow \frac{99}{100} = \sqrt{\left(1-\frac{V^2}{c^2}\right)} \quad \Rightarrow 1-\frac{V^2}{c^2} = \left(\frac{99}{100}\right)^2$$

$$\Rightarrow 1-\frac{V^2}{c^2} = = \frac{9801}{10000} \Rightarrow \frac{V^2}{c^2} = \left(\frac{199}{10000}\right)$$

$$\Rightarrow V = \left(\frac{\sqrt{199}}{100}c\right) = \left(0.423\times10^8\right)ms^{-1}$$

27. The proper life of meson is $2x10^{-8}$sec. calculate the mean life of meson when it moves with a velocity 0.8c?

Sol: We have $\therefore\ \Delta t^1 = \frac{\Delta t}{\sqrt{1-\frac{V^2}{c^2}}} \Rightarrow \Delta t^1 = \frac{2\times10^{-8}}{\sqrt{1-\frac{(0.8c)^2}{c^2}}}$

$$= \frac{2\times10^{-8}}{\sqrt{1-0.64}} = \frac{2\times10^{-8}}{\sqrt{0.36}} = \frac{2\times10^{-8}}{0.6} = 3.33\times10^{-8}\ \text{sec}$$

28. The proper life of meson is 2×10^{-8}sec, when it moves with a velocity of 2.4×10^{10}cm s^{-1} . Calculate distance travelled by it before disintegrating and distance it would travel if there were no relative effect.

Sol: We have $\therefore\ \Delta t^1 = \frac{\Delta t}{\sqrt{1-\frac{V^2}{c^2}}}$

$$\Rightarrow \Delta t^1 = \frac{2\times10^{-8}}{\sqrt{1-\frac{\left(2.4\times10^{10}\right)^2}{\left(3\times10^{10}\right)^2}}} = 4.17\times10^{-8}\ \text{sec}$$

Distance travelled=velocity $\times$ time= $\left(2.4\times10^{10}\right)\times$ $\left(4.17\times10^{-8}\right) = 1000\,cm$

Distance travelled if there were no relative effect = $\left(2.4\times10^{10}\right)\times\left(2.5\times10^{-8}\right)$ cm

29. *What is the velocity of π mesons whose observed life is 2.5 x10^{-7}sec. The proper life of meson is 2.5× 10^{-8}sec.*

Sol: We have, $\Delta t^1 = \dfrac{\Delta t}{\sqrt{1-\dfrac{V^2}{c^2}}}$

$$\Rightarrow 2.5\times10^{-7} = \frac{2.5\times10^{-8}}{\sqrt{1-\dfrac{V^2}{c^2}}} \Rightarrow 10 = \frac{1}{\sqrt{1-\dfrac{V^2}{c^2}}}$$

$$\Rightarrow 1-\frac{V^2}{c^2} = \frac{1}{100} \Rightarrow \frac{V^2}{c^2} = \frac{99}{100}$$

$$V^2 = \frac{99}{100}c^2 \Rightarrow V = \frac{\sqrt{99}}{10}c = 0.995c = 2.9849\times10^{-8}\text{ sec}$$

30. *A clock shows correct time. With what speed it should be moved relative to an observer so that it may appear to lose 4 minutes in 24 hours.*

Sol: We have $\Delta t^1 = \dfrac{\Delta t}{\sqrt{1-\dfrac{V^2}{c^2}}}$ here $\Delta t = 24\times60 = 1440\text{ min}$

$$\Delta t^1 = 1440+4 = 1444\text{ min}$$

$$\Rightarrow 1444 = \frac{1440}{\sqrt{1-\dfrac{V^2}{c^2}}} \Rightarrow \sqrt{1-\frac{V^2}{c^2}} = \frac{1440}{1444} \Rightarrow \sqrt{1-\frac{V^2}{c^2}} = 0.9972$$

$$\Rightarrow 1-\frac{V^2}{c^2} = 0.9944 \Rightarrow \frac{V^2}{c^2} = 0.0056$$

$$\Rightarrow V = 0.7483c = \left(2.23\times10^7\right)m\sec^{-1}$$

31. *Two particles came towards each other with a speed of 0.8 with respect to laboratory. Find their relative speed.*

Sol: We have $u = \dfrac{u^1+V}{1+\dfrac{u^1 V}{c^2}} = \dfrac{0.8c+0.8c}{1+\dfrac{0.8c\times0.8c}{c^2}}$

$$= \frac{1.6c}{1+(0.8\times0.8)} = \frac{1.6c}{1.64} = 0.975c$$

Thus the relative speed is 0.975c.

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UNIT-IV

5. UNDAMPED, DAMPED & FORCED OSCILLATIONS

ESSAY QUESTIONS

1. *Define Simple Harmonic Motion. Give its Characteristics.*

Ans: **Simple Harmonic Motion:** This is a special type of periodic motion in which the body moves along a straight line such that its acceleration is always directed towards a fixed point, and is directly proportional to its displacement but opposite in direction.

Characteristics of simple harmonic motion:

a) The motion is periodic.

b) The motion is along a straight line about the mean position.

c) The acceleration is proportional to displacement and in opposite direction.

d) Acceleration is always directed towards the mean position.

2. *Derive the Equation of motion of a simple harmonic oscillator and its solution.*

Ans: Consider a particle 'P' of mass m executing S.H.M. about equilibrium position 'O' along X-axis. Let x be the displacement of P from O at any instant. The instantaneous force F acting upon P is given by,

$$F \alpha - x \Rightarrow F = -k x \quad \ldots (1)$$

Where k is proportionality constant. The negative sign is used to show that the force F is opposite to the displacement.

According to Newton's second law of motion,

$$\text{F=ma} \Rightarrow F = m\left(\frac{d^2x}{dt^2}\right) \quad \ldots (2)$$

From eqs. (1) and (2): $m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$

$$\therefore \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{Where } \frac{k}{m} = \omega^2$$

This is the differential equation of simple harmonic oscillator.

Solution for the equation: The simple harmonic oscillator equation is $\frac{d^2x}{dt^2} + \omega^2 x = 0$ … (1)

Let us assume a trial solution of the form, $x = c e^{\alpha t}$ Where C and α are arbitrary constants.

Differentiating the eqn, (1) we get $\frac{dx}{dt} = c\alpha e^{\alpha t}$

$$\Rightarrow \frac{d^2x}{dt^2} = c\alpha^2 e^{\alpha t}$$

Substituting these values in the equation (1):

$$\Rightarrow c\alpha^2 e^{\alpha t} + \omega^2 c e^{\alpha t} = 0$$

$$\Rightarrow c\,e^{\alpha t}(\alpha^2+\omega^2)=0 \;\; \Rightarrow(\alpha^2+\omega^2)=0$$

$$(\because c\,e^{\alpha t}=0)$$

$$\therefore \alpha=\pm\sqrt{(-\omega^2)}=\pm i\omega \quad \text{Where } i=\sqrt{-1}$$

So the general solution of the equation (1) is

$$x=c_1\,e^{+i\omega t}+c_2\,e^{-i\omega t}$$

Where C_1 and C_2 are arbitrary constants.

Further, $x=\; C_1(\cos\omega t+i\sin\omega t)+C_2(\cos\omega t-i\sin\omega t)$

$$x=\;(C_1+C_2)\cos\omega t+i(C_1-C_2)\sin\omega t$$

Let us put, $(C_1+C_2)=a\sin\phi$ and $i(C_1-C_2)=a\ cos\,\phi$

Where a and ϕ are new constants.

$$\therefore\; x=a\,\sin\phi\,\cos\omega\,t+a\,\cos\,\phi\,\sin\omega t$$

$$\Rightarrow x=a\,\sin\left(\omega\,t+\phi\right)$$

This is the solution of the equation of simple harmonic oscillator.

3. *Define damping. Derive theequation of damped harmonic oscillator and explain over damped, critically damped, under damped motions.*

Ans: Damping: For an ideal harmonic oscillator, the amplitude of vibration remains constant. When a body vibrates in a medium which offers resistance to its motion, the amplitude of vibration decreases gradually and finally the body comes to rest. This is due to the resistance offered by the medium. Then the motion of the body is known as damped harmonic motion. This phenomenon is called damping

Example: If we displace a pendulum from its equilibrium position it will oscillate with decreasing amplitude and finally comes to rest in equilibrium position.

Equation of damped harmonic oscillator: Consider a damped harmonic oscillator of mass 'm'. When it is in motion, the forces acting on it are,

1. **Restoring force:** The restoring force is proportional to displacement but oppositely directed.

   $F_1\,\alpha-x\Rightarrow F_1=-kx$, Where k is a constant of proportionality or force constant.

2. **Resistive force:** The resistive force is proportional to velocity but oppositely directed.

   $F_2\,\alpha-\dfrac{dx}{dt}\Rightarrow F_2=-\,r\dfrac{dx}{dt}$, Where r is the frictional force per unit velocity.

Resultant force F = mass x acceleration = $m\dfrac{d^2x}{dt^2}$

$$F=F_1+F_2 \;\Rightarrow\frac{d^2x}{dt^2}+\frac{r}{m}\frac{dx}{dt}+\frac{k}{m}x=0$$

$$\Rightarrow\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\omega^2x=0 \qquad \ldots(1)$$

where $\dfrac{r}{m}=2b,\ \omega^2=\dfrac{k}{m}$

This is the differential equation of damped harmonic oscillator.

Solution for the equation: Eq. (1) is a differential equation of second order.

Let its solution be $x=Ae^{\alpha t}$ … (2)

where A and α are arbitrary constants.

Differentiating equation (2) with respect to t, we get

$$\frac{dx}{dt}=A\alpha e^{\alpha t}\;\; and \;\; \frac{d^2x}{dt^2}=A\alpha^2e^{\alpha t}$$

Substituting these values in eq. (1), we have,

$$(1) \Rightarrow A\alpha^2 e^{\alpha t} + 2bA\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

$$\Rightarrow Ae^{\alpha t}\left(\alpha^2 + 2b\alpha + \omega^2\right) = 0$$

$$\Rightarrow \left(\alpha^2 + 2b\alpha + \omega^2\right) = 0 \quad \left(\because Ae^{\alpha t} \neq 0\right)$$

$$\Rightarrow \alpha = -b \pm \sqrt{b^2 - \omega^2}$$

The general solution of eq. (1) is given by

$$x = A_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + A_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t} \quad \ldots (3)$$

Where A_1 and A_2 are arbitrary constants.

Case (1): Over damped motion: When $b^2 > \omega^2$

In this case $\left(b^2 - \omega^2\right)$ is real and less than b. Now the powers $\left(-b - \sqrt{b^2 - \omega^2}\right)$ and $\left(-b + \sqrt{b^2 - \omega^2}\right)$ in eq. (3) are both negative. Thus the displacement *x* consists of two terms, both exponentially decreases to zero. In this case the body once displaced returns to its equilibrium position very slowly without performing any oscillation. This type of motion is called as over-damped or dead beat.

Ex: This type of motion is shown by a pendulum moving in thick oil or by a dead beat moving coil galvanometer

Case (2): Critical Damping: when $b^2 = \omega^2$

If we put $b^2 = \omega^2$ in equation (3), then this solution does not satisfy the differential equation (1). Let us consider that $\sqrt{b^2 - \omega^2}$ is not zero but this is equal to a very small quantity 'h'.

i.e., $\sqrt{b^2 - \omega^2} = h \rightarrow 0$. Now eq. (3) reduces to

$$x = A_1 e^{(-b+h)t} + A_2 e^{(-b-h)t} = e^{-bt}\left(A_1 e^{ht} + A_2 e^{ht}\right)$$

$$= e^{-bt}\left(A_1\left(1 + ht + \ldots\right) + A_2\left(1 - ht + \ldots\right)\right)$$

$$= e^{-bt}\left[\left(A_1 + A_2\right) + ht\left(A_1 - A_2\right) + \ldots.\right]$$

($\because$ his very small higher powers are neglected)

$$= e^{-bt}\left(p + qt\right) \quad \ldots (4)$$

Where $p = \left(A_1 + A_2\right)$ and $q = h\left(A_1 - A_2\right)$

Eq. (4) represents a possible form of a solution. It is clear from eq. (4) that as t increases, the factor (p+qt) increases but the factor e^{-bt} decreases. In this way the displacement term e^{-bt} and the displacement approaches zero as t increases. In this case the particle tends to acquire its position of equilibrium much rapidly than in case (1). Such a motion is called critical damped motion.

Ex: Motion of pointer in the instruments such as voltmeter, ammeter etc., In this the pointer moves to correct position and comes to rest without any oscillation in minimum time.

Case (3): Under damped motion: when $b^2 < \omega^2$, In this case $\sqrt{b^2 - \omega^2}$ is imaginary.

Let us write $\sqrt{b^2 - \omega^2} = i\sqrt{\omega^2 - b^2} = i\beta$

where $\beta = \sqrt{\omega^2 - b^2}$ *and* $i = \sqrt{(-1)}$

$$\therefore (3) \Rightarrow x = A_1 e^{(-b+i\beta)t} + A_2 e^{(-b-i\beta)t} = e^{-bt}\left(A_1 e^{i\beta t} + A_2 e^{i\beta t}\right)$$

$$= e^{-bt}\left(A_1\left(\cos\beta t + i\sin\beta t\right) + A_2\left(\cos\beta t - i\sin\beta t\right)\right)$$

$$= e^{-bt}\left[\left(A_1 + A_2\right)\cos\beta t + i\left(A_1 - A_2\right)\sin\beta t\right]$$

$$= e^{-bt}\left[a\sin\phi\cos\beta t + a\cos\phi\sin\beta t\right]$$

where $a\sin\phi = \left(A_1 + A_2\right), a\cos\phi = \left(A_1 - A_2\right)$

$$x = e^{-bt} a \sin(\beta t + \phi) = ae^{-bt} \sin\left[\sqrt{(\omega^2 - b^2)}t + \phi\right] \quad \ldots(5)$$

This equation represents the simple harmonic motion with amplitude ae^{-bt} and time period.

$T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{(\omega^2 - b^2)}}$ The amplitude of the motion is continuously decreasing owing to the factor e^{-bt} which is called damping factor. The amplitude varies between ae^{-bt} and $-ae^{-bt}$. The decay of the amplitude depends upon the damping coefficient 'b'. It is called "under damped" motion.

Ex:The motion of a pendulum in air, the motion of the coil of ballistic galvanometer or the electric oscillations of LCR circuit.

4. Explain theMethods for the estimation of damping.

Ans: There are 3 methods for the estimation of damping. They are:

Logarithmic decrement: Logarithmic decrement measures the rate at which the amplitude decreases. The amplitude of damped harmonic oscillator is given by, amplitude $= ae^{-bt}$.

At t=0, amplitude $a_0 = a$

Let $a_1, a_2, a_3 \ldots.$ be the amplitudes at time t = T, 2T, 3T …respectively, where T= period of oscillation. Then, $a_1 = ae^{-bT}$, $a_2 = ae^{-b(2T)}$, $a_3 = ae^{-b(3T)} \ldots$

From these equations we get, $\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots\ldots = e^{bT}$

$= e^{\lambda} (where\ \ bT = \lambda)$

Taking the natural logarithm, we get,

$$\log_e e^{\lambda} = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} \ldots\ldots.$$

$$\Rightarrow \lambda = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} \ldots..$$

Where λ is known as logarithmic decrement.

Thus logarithmic decrement is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are separated by one period.

Relaxation time:The relaxation time is defined as the time taken for the total mechanical energy to decay to $(1/e)^{th}$ of its original value.

The mechanical energy of damped harmonic oscillator is giveby. $E = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} m\omega^2 a^2 e^{-2bt} \left(\because A = a e^{-bt}\right)$

Let $E = E_0\ when\ t = 0,$

$$\therefore\ E_0 = \frac{1}{2} m\omega^2 a^2 \quad Now\, E = E_0 e^{-2bt} \quad \ldots (1)$$

Let τ be the relaxation time, i.e. at t=τ, $E = \frac{E_0}{e}$

$$\frac{E_0}{e} = E_0\, e^{-2b\tau} \Rightarrow e^{-1} = e^{-2b\tau} \Rightarrow 2b\tau = 1 \Rightarrow \tau = \frac{1}{2b} \quad \ldots (2)$$

From eqs. (1) and (2) we get $E = E_0 e^{\frac{-t}{\tau}}$ … (3)

The expression of power dissipation can be written as

$$P = \frac{E}{\tau} \quad \ldots (4)$$

Quality factor: The quality factor is defined as 2π times the ratio of the energy stored in the system and the energy lost per period.

$$Q = 2\pi \frac{energy\ stored\ in\ system}{energy\ lost\ per\ period} = 2\pi \frac{E}{PT} \quad \ldots (5)$$

Where P is the power dissipatedand T is periodic time. We know that $P = \frac{E}{\tau}$

Where τ is the relaxation time . So,

$$Q = 2\pi \frac{E}{\left(\frac{E}{\tau}\right)T} = \frac{2\pi\tau}{T} \quad \therefore Q = \omega\tau \quad \ldots (6)$$

where $\omega = \frac{2\pi}{T}$ = angular frequency.

5. *Define natural and forced vibrations. Derive theequation of damped harmonic oscillator and arrive the condition for resonance.*

Sol: Natural or free vibrations: When a body is made to vibrate and left free itself it always vibrates with a frequency known as natural frequency. Those vibrations are called natural or free vibrations.

Forced vibrations: When the body vibrates with a frequency other than its natural frequency under the action of an external periodic force, then those vibrations are called Forced vibrations

Equation of forced oscillator: Consider a forced oscillator of mass 'm'. When it undergoes forced oscillations the forces acting on it are,

1) ***Restoring force (F_1):*** The restoring force proportional to the displacement but oppositely directed, F_1= -ky, where k is known as force constant.
2) ***Resistive force (F_2):*** The resistive force is proportional to velocity but oppositely directed.
   $F_2\, \alpha - \frac{dx}{dt} \Rightarrow F_2 = -r\left(\frac{dx}{dt}\right)$ Where r is the resistive force per unit velocity.
3) ***External periodic force (F3) :*** It is represented by $F_3 = F \sin pt$. Where F is the maximum value of this force and $\frac{p}{2\pi}$ is its frequency. So the total force acting on the particle is given by, $F = F_1 + F_2 + F_3$

$$\Rightarrow F = -kx - r\frac{dx}{dt} + F\sin pt$$

By Newton's second law of motion $F = m\frac{d^2x}{dt^2}$

$$\therefore\ m\frac{d^2x}{dt^2} = -kx - r\frac{dx}{dt} + F\sin p$$

$$\Rightarrow m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F\sin pt$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}\sin pt$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f \sin pt \qquad \ldots (1)$$

Where $\frac{r}{m} = 2b, \frac{k}{m} = \omega^2 \; and \; \frac{F}{m} = f$

Eq. (1) is the differential equation of forced oscillator. The solution of differential equation (1) must be of the type $x = A\sin(pt-\theta)$ A and θ are arbitrary constants.

Differentiating eq. (2) we have $\frac{dx}{dt} = A\cos(pt-\theta)$ and

$$\frac{d^2x}{dt^2} = -A\,p^2 \sin(pt-\theta)$$

Substituting these values in eq. (1) we get:

$$-A\,p^2 \sin(pt-\theta) + 2bAp\cos(pt-\theta) + \omega^2 A\sin(pt-\theta)$$

$$= f \sin pt = f \sin\{(pt-\theta)+\theta\}$$

$$A(\omega^2 - p^2)\sin(pt-\theta) + 2bAp\cos(pt-\theta)$$

$$= f\sin(pt-\theta)\cos\theta + f\cos(pt-\theta)\sin\theta$$

Comparing the coefficients of $\cos(pt-\theta)$ and $\sin(pt-\theta)$ on both sides,

$$A(\omega^2 - p^2) = f\cos\theta \qquad \ldots (3)$$

$$\text{and } 2bAp = f\sin\theta \qquad \ldots (4)$$

Squaring equations (3) and (4) and then adding, we get

$$A^2(\omega^2 - p^2)^2 + 4b^2A^2p^2 = f^2$$

$$A^2[(\omega^2 - p^2)^2 + 4b^2p^2] = f^2$$

$$\therefore A = \frac{f}{\sqrt{[(\omega^2 - p^2)^2 + 4b^2p^2]}} \qquad \ldots (5)$$

$$\frac{(4)}{(3)}: \; \tan\theta = \frac{2bAp}{A(\omega^2 - p^2)} = \frac{2bp}{(\omega^2 - p^2)}$$

$$\Rightarrow \theta = Tan^{-1}\left(\frac{2bp}{\omega^2 - p^2}\right) \qquad \ldots (6)$$

Substituting the value of A from Eq. (5) in Eq. (2), we get

$$x = \frac{f}{\sqrt{[(\omega^2 - p^2)^2 + 4b^2p^2]}}\sin(pt-\theta) \qquad \ldots (7)$$

Eq. (5) gives the amplitude of forced vibration while (6) gives its phase.

Resonance: when a body vibrates under the action of an external periodic force, whose frequency is equal to the natural frequency of the vibrating body, the amplitude becomes maximum. This phenomenon is called resonance and those vibrations are called resonant vibrations.

Condition for amplitude resonance:The amplitude of forced oscillations varies with the frequency of applied force and becomes maximum at a particular frequency. This phenomenon is known as amplitude resonance

In this case of forced vibrations, we have

$$A = \frac{f}{\sqrt{[(\omega^2 - p^2)^2 + 4b^2p^2]}}\sin(pt-\theta) \qquad \ldots (1)$$

and $\theta = Tan^{-1}\left(\frac{2bp}{\omega^2 - p^2}\right)$.

The expression (1) shows that the amplitude varies with the frequency of the force p. For a particular value of p, the amplitude becomes maximum. The phenomenon is called Amplitude Resonance. The amplitude is maximum when $\sqrt{[(\omega^2 - p^2)^2 + 4b^2p^2]}$ is minimum.

$$\Rightarrow \frac{d}{dp}[(\omega^2 - p^2)^2 + 4b^2p^2] = 0$$

$$\Rightarrow 2(\omega^2 - p^2)(-2p) + 4b^2(2p) = 0$$

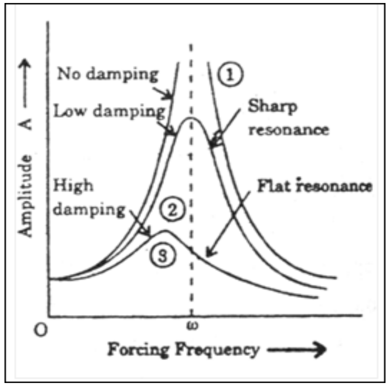
$$\Rightarrow (\omega^2 - p^2) = 2b^2$$

$$\Rightarrow p = \sqrt{(\omega^2 - 2b^2)} \quad \ldots (3)$$

If the damping is small, the value of 'b' can be neglected and the condition of maximum amplitude reduced to $p = \omega$.

6. *Explain the Sharpness of resonance.*

Sol: We have seen that amplitude of the forced oscillation is maximum when the frequency of the applied force has a value to satisfy the condition of resonance i.e. $p = \sqrt{(\omega^2 - 2b^2)}$. If the frequency changes from this value, the amplitude falls. When the fall in amplitude is very large, for a small change in the resonance condition the resonance is said to be sharp. If the fall in amplitude is small, the resonance is termed as flat. Thus the term sharpness of resonance means the rate of fall in amplitude with the change of forcing frequency on each side of resonance frequency.



The graph shows the variation of amplitude with forcing frequency at different amounts of damping. It is observed from the figure that, the resonance is sharp for smaller damping (curve-2) and the resonance is flat when damping is

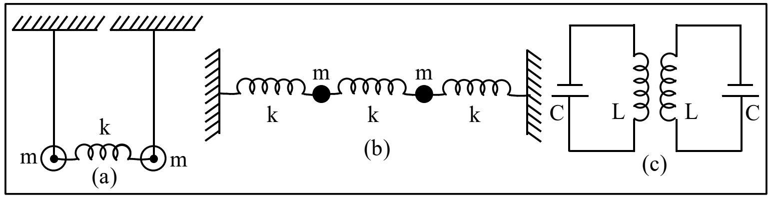
large(curve-3). Hence, smaller is damping, sharper is resonance or larger is the damping, flatter is the resonance. When there is no damping (curve-1) the amplitude becomes infinity.

6. COUPLED OSCILLATIONS

7. *What are coupled oscillators? What are normal coordinates and normal modes.*

Ans: Coupled oscillators :When the oscillations ofthe oscillators are coupled with one another, they are called coupled oscillators. In this the motion of one oscillator is effected by the other.

Ex: Two coupled oscillating systems are shown in Fig. The fig (a) shows two simple pendulums with their bobs connected to each other by a spring. Fig (b) shows two masses attached to each other by three springs. The middle spring provides the coupling. Fig (c) shows the coupled LC circuits.



Normal coordinates and Normal modes: The *normal coordinate* of a coupled system are the parameters in terms of which the equations of motion of the system can be expressed as a set of linear differential equations with constant coefficients in which each equation contains only one dependent variable.

The S.H.M. associated with each normal coordinate is called a *normal mode* of the coupled system.

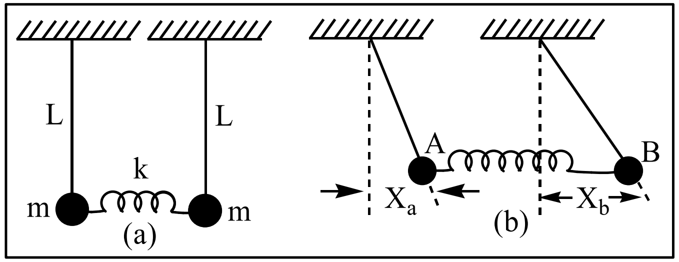
A two coupled oscillator has two normal modes, as in phase mode and an out of phase mode. The in phase mode has a frequency equal to the natural frequency of either of the oscillators. The out of phase mode has a frequency slightly greater than the natural frequency of the oscillators.

Significance: the normal modes of vibration are entirely independent of each other. The energy associated with the normal mode is never exchanged with another mode. So we can add the energies of the separate modes to get total energy.

The general motion of any coupled system can always be represented as a superposition of all possible normal modes.

8. Explain the theory of two coupled oscillators its normal coordinates and normal modes.

Ans: Two coupled Pendulums: Consider a system of two identical simple pendulums A and B each of mass *m* and length *l*, coupled by a linear spacing of force constant k. The separation between the bobs is such that the spring is relaxed in the equilibrium position (Fig .a)



The system is disturbed slightly from its equilibrium position as in above fig.(b) and released. The two pendulums begin to oscillate. Let x_a and x_b be the displacements of bobs A and B at an instant of time t respectively.

When$x_b > x_a$ the spring is stretched and compressed.

When $x_a > x_b$, the magnitude of the tension in the spring is $k(x_b - x_a)$. This tension will act along the direction of the restoring force mg $\sin\theta = mg\frac{x_b}{l}$ of the pendulum B but in the opposite direction of restoring force mg $\frac{x_a}{l}$ of the pendulum A. The equations of the motion of pendulums A and B for small oscillations in a planeare given by,

$$m\frac{d^2x_a}{dt^2} = -\frac{mg}{l}x_a + k(x_b - x_a) \qquad \dots (1)$$

And $$m\frac{d^2x_b}{dt^2} = -\frac{mg}{l}x_b - k(x_b - x_a) \qquad \dots (2)$$

These equations are not of S.H.M. as the acceleration of the pendulum is not proportional to its own displacement. In the absence of spring i.e., k = 0, the two pendulums will execute simple harmonic oscillations, whose angular frequency ω_0 is given by, $\omega_0 = \sqrt{\frac{g}{l}}$

In terms of ω_0, the two coupled Eq. (1) and (2) can be written as,

$$\frac{d^2x_a}{dt^2} = -\omega_0^2 x_a + \frac{k}{m}(x_b - x_a) \qquad \dots (4)$$

and $\frac{d^2x_b}{dt^2} = -\omega_0^2 x_b - \frac{k}{m}(x_b - x_a)$ … (5)

solving the equations for x_a and x_b:

$$(4)+(5): \quad \frac{d^2}{dt^2}(x_a + x_b) = -\omega_0^2(x_a + x_b)$$

$$\Rightarrow \frac{d^2}{dt^2}(x_a + x_b) + \omega_0^2(x_a + x_b) = 0 \quad \ldots (6)$$

Subtracting Eq. (1) from Eq. (2),

$$(5) - (4): \quad \frac{d^2}{dt^2}(x_b - x_a) = -\omega_0^2(x_b - x_a) - 2\frac{k}{m}(x_b - x_a)$$

$$\Rightarrow \frac{d^2}{dt^2}(x_b - x_a) + \left(\omega_0^2 + 2\frac{k}{m}\right)(x_b - x_a) = 0 \quad \ldots (7)$$

Eq. (6) and (7) and familiar equations of S.H.M. In Eq. (6) the variable is $(x_a + x_b)$ and in eq. (7), the variable is $(x_b + x_a)$.

If $x_a = x_b$ at all times, the motion is completely described by Eq. (6). The angular frequency of oscillation is given by

$$\omega_1 = \omega_0 = \sqrt{\frac{g}{l}} \quad \ldots (8)$$

This frequency is the same as that of the angular frequency of either pendulum when they are isolated i.e., the effect of spring is absent. This is due to the fact that both pendulums are in phase always and the spring has the natural length throughout its motion.

If $x_a = -x_b$ always, the motion is completely described by Eq. (7). In which case, the angular frequency is given by

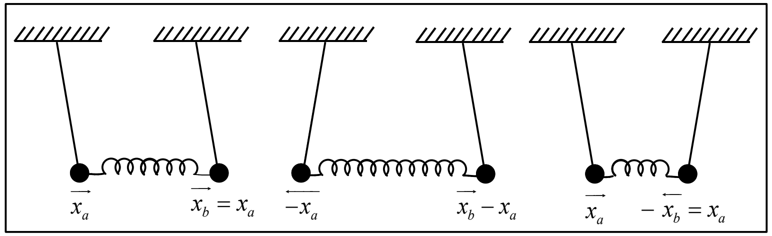
$$\omega_2 = \sqrt{\left(\omega_0^2 + 2\frac{k}{m}\right)} \quad \ldots (9)$$

From eq. (8) and (9), it is clear that $\omega_2 > \omega_1$. Hence the two pendulums are oscillating harmonically with a frequency ω_2. They are always out of phase.

Thus during this motion, the spring is extended and also compressed. Hence the coupling is effective in this case.

Normal coordinates and Normal modes : The normal coordinates of a coupled systems are the parameters in terms of which the equations of motion of the system can be expressed as a set of linear differential equations with constant coefficients in which each equation contains only one dependent variable.

The simple harmonic motion associated with each normal coordinate is called a normal mode of the coupled system. Each normal mode has its own characteristic frequency, called the normal mode frequency. In the case of two coupled pendulums, normal mode frequencies are ω_1 and ω_2. In the first mode, both pendulums are displaced through the same distance from the equilibrium position in the same direction so that $\frac{x_b}{x_a} = l$. When they are released, each executes S.H.M. with the same frequency $\omega_1 = \sqrt{\frac{g}{l}}$ They are always in phase and $\frac{x_b}{x_a} = l$ throughout its motion. This is called in phase mode of the system.



In the second mode, both pendulums are displaced through the same distance from equilibrium position in opposite directions, so that $\frac{x_a}{x_b} = -1$. When they are released, each pendulum executes S.H.M. with frequency.

$$\omega_2 = \sqrt{\omega_0^2 + \frac{2k}{m}} = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

They are always out of phase, and $\frac{x_b}{x_b} = -l$ through out the motion. In this case, both the pendulums cross their equilibrium position simultaneously. This is called out of phase mode of the system.

Normal Modes Solution : The equations governing the motion of the pendulum are

$$\frac{d^2x_a}{dt^2} = -\omega_0^2 x_a + \frac{k}{m}\left(x_b - x_a\right)$$

and $\frac{d^2x_b}{dt^2} = -\omega_0^2 x_b - \frac{k}{m}\left(x_b - x_a\right)$ Eq. (4) & (5)

The above differential equations are solved to obtain normal modes. Consider that a normal mode exists at an angular frequency w and phase constant f. This means that

both the pendulums move with a S.H.M. with the same angular frequency w and with the same phase constant f. They

$$x_a = C\cos\left(\omega t + f\right) \qquad \ldots (10)$$

and $x_b = C^1 \cos\left(\omega t + f\right)$ … (11)

where C and C^1 are the amplitudes which may be different. From Eq. (10) and (11), we get

$$\frac{d^2x_a}{dt^2} = -\omega^2 C\cos\left(\omega t + \phi\right) = -\omega^2 x_a$$

and $\frac{d^2x_b}{dt^2} = -\omega^2 \cos\left(\omega t + \phi\right) = -\omega^2 x_b$

Hence $-\omega^2 x_a = \omega_0^2 x_a + \frac{k}{m}\left(x_b - x_a\right)$

$$\Rightarrow \left(\omega_0^2 - \omega^2 + \frac{k}{m}\right)x_a = \frac{k}{m}x_b \qquad \ldots (12)$$

And $\left(\omega_0^2 - \omega^2 + \frac{k}{m}\right)x_b = \frac{k}{m}x_a$ … (13)

From Eq. (12), we get

$$\frac{x_a}{x_b} = \frac{\left(\frac{k}{m}\right)}{\left[\omega_0^2 - \omega^2 + \left(\frac{k}{m}\right)\right]} \qquad \ldots (14)$$

Similarly, from Eq. (13), we get

$$\frac{x_a}{x_b} = \frac{\left[\omega_0^2 - \omega^2 + \left(\frac{k}{m}\right)\right]}{\left(\frac{k}{m}\right)} \qquad \ldots (15)$$

From Eq. (14) and (15), we get

$$\frac{\left(\frac{k}{m}\right)}{\left[\omega_0^2-\omega^2+\left(\frac{k}{m}\right)\right]}=\frac{\left[\omega_0^2-\omega^2+\left(\frac{k}{m}\right)\right]}{\left(\frac{k}{m}\right)}$$

$$\Rightarrow\left[\omega_0^2-\omega^2+\left(\frac{k}{m}\right)\right]^2=\left(\frac{k}{m}\right)^2$$

$$\Rightarrow\left[\omega_0^2-\omega^2+\left(\frac{k}{m}\right)\right]=\frac{k}{m}$$

Thus we have two solutions for w which may be denoted by w and w.

$$\therefore\ \omega^2=\omega_0^2 \qquad \ldots (16)$$

$$\text{and } \omega^2=\omega_0^2+2\left(\frac{k}{m}\right) \qquad \ldots (17)$$

The +ve roots of Eq. (16) and (17) are the two normal frequencies of the system i.e., of the two modes. The angular frequency of mode 1 is ω while that of mode 2 is $\omega^{||}$.

The configuration of mode 1 can be obtained by substituting $\omega_0^2=\omega^2$ in Eq. (14) or (15). Thus, we get

$$\left(\frac{x_a}{x_b}\right)_{\text{mode 1}}=1 \text{ and } \left(\frac{C^1}{C}\right)_{\text{mode 1}}=1 \text{ from Eq. (10) \& (11)}$$

$$\text{Hence } \left(\frac{x_a}{x_b}\right)_{\text{mode 1}}=\left(\frac{C^1}{C}\right)_{\text{mode 1}}=1 \qquad \ldots (18)$$

The displacement of oscillators in mode 1 are given by

$$(x_a)_1=C\cos(\omega t+f_1) \qquad \ldots (19)$$

$$\text{and } (x_b)_1=C\cos(\omega t+f_1) \qquad \ldots (20)$$

The displacement of oscillators in mode 2 are given by

$$(x_a)_2=D\cos(\omega^{||}t+f_2) \qquad \ldots (21)$$

$$\text{and } (x_b)_2=-D\cos(\omega^{||}t+f_2) \qquad \ldots (22)$$

The most general solution is given by the superposition of the two normal modes.

$$x_a=(x_a)_1+(x_b)_2$$
$$=C\cos(\omega t+f_1)+D\cos(\omega^{||}t+f_2)$$
$$x_b=(x_b)_1+(x_b)_2$$
$$=C\cos(\omega t+f_1)-D\cos(\omega^{||}t+f_2)$$

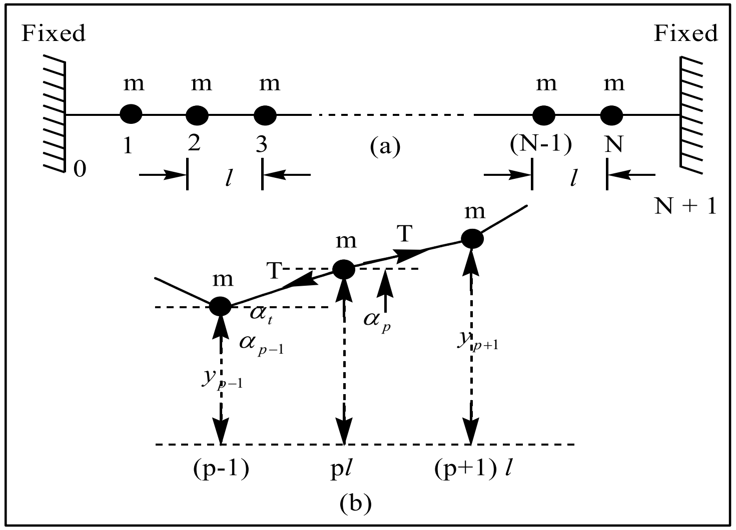
These are known as normal modes.

9. *Explain the theory of N-coupled oscillators.*

Ans: N-coupled Oscillators : Consider a flexible elastic string of negligible mass supporting ***n*** identical particles each of mass *m* equally spaced at distances *l* along its length as shown in Fig. (a). The string is fixed at both ends, so that a constant tension T is present at all points and at all times in the string. The masses are located at distances x = *l*, 2*l*, 3*l*, …. N*l*. The total length of the string is (N + 1) *l*. The two particles at the fixed ends are considered as if they were particles of zero displacement.

Fig. (b) shows the configuration of (p – 1), p and (p + 1)th particles at some instant time during their transverse oscillation. Here it is assumed that the amplitude of these oscillations is very small, so that the initial tension t in the string does not change as the particle oscillate.

Equation of Motion: Fig. (b) represents the configuration of the particles at a certain instant of time during their transverse oscillation. Consider successive particles indicated as the p^th^ particle together with two of its immediate neighbours (p – 1)th and (p + 1)th particles. Let their displacements from the equilibrium position be $y_p, y_{(p-1)}$ and $y_{(p+1)}$ respectively, where p = 1, 2, 3 …. (N – 1), N.



The resultant y – component of force on the p^th^ particle is given by

$$F_p = -T\sin a_{p-1} + T\sin a_p \qquad \text{As a is small, } \sin a = \tan a$$

$$F_p = -T\sin a_{p-1} + T\tan a_p \qquad \ldots (1)$$

From Fig. (b), we have

$$\tan\alpha_p = \frac{y_{p+1} - y_p}{l} \text{ and } \tan\alpha_{p-1} = \frac{y_p - y_{p-1}}{l}$$

Substituting the above values in Eq. (1), we have

$$F_p = -\frac{T}{l}\left(y_p - y_{p-1}\right) + \frac{T}{l}\left(y_{p+1} - y_p\right)$$

$$= \frac{T}{l}\left(y_{p+1} + y_{p-1} - 2y_p\right)$$

This force must be equal to mass *m* times the transverse acceleration of the p^th^ particle. Thus the equation of motion of the p^th^ particle can be expressed as,

$$m\frac{d^2y_p}{dt^2} = \frac{T}{l}\left(y_{p+1} + y_{p-1} - 2y_p\right)$$

$$\Rightarrow \frac{d^2y_p}{dt^2} = \frac{T}{ml}\left(y_{p+1} + y_{p-1} - 2y_p\right) \ldots (2)$$

This is the differential equation for the p^th^ considered particle. By putting p = 1, 2, 3, … N, we have a set of N differential equations.

Here we have the following two boundary conditions, viz.,

$$x = 0 \qquad y_0 = 0$$

$$\text{and} \quad x = (N+1)l \qquad y_{N+1} = 0 \qquad \ldots (3)$$

Notmal Modes : For normal modes, Let there may be a mode with angular frequency w and phase constant f. In normal mode, all particles execute simple harmonic oscillations with the same frequency w and constant phase f. Thus for the p^th^ particle,

$$y_p = A_p\cos(\omega t + 1) \qquad \ldots (4)$$

where A_p is the amplitude of simple harmonic oscillations of the p^th^ particle.

Similarly,

$$Y_{p-1} = A_{p-1}\cos(\omega t + f) \text{ and } \quad Y_{p+1} = A_{p+1}\cos(\omega t + f)$$

From Eq. (4), we have $\frac{d^2 y_p}{dt^2} = -\omega^2 A_p \cos(\omega t + f)$

Substituting this value in Eq. (2), we have

$$-\omega^2 \cos(\omega t + f) = \frac{T}{ml}\left(A_{p+1} + A_{p-1} - 2A_p\right)\cos(\omega t + f)$$

$$\Rightarrow \quad \omega^2 A_p = \frac{T}{ml}\left(A_{p+1} + A_{p-1} - 2A_p\right)$$

$$\Rightarrow \quad A_{p+1} + A_{p-1} = \left(2 - \frac{\omega^2 ml}{T}\right)A_p \qquad \ldots (5)$$

Applying boundary conditions $A_0 = 0$ and $A_{p+1} = 0$, in Eq. (5) represents a set of N equation which have to be simultaneously solved to get the possible mode of frequencies.

General solution : EQ. (5) can be rewritten in the following form :

$$\frac{A_{p+1} + A_{p-1}}{A_p} = 2 - \frac{\omega^2 ml}{T} \text{ Put } \frac{T}{ml} = \omega_0^2$$

$$\therefore \frac{A_{p+1} + A_{p-1}}{A_p} = 2 - \frac{\omega^2}{\omega_0^2} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \qquad \ldots (6)$$

Here p = 1, 2, 3 …. N. For any particular value of w, the R.H.S. of Eq. (6) is constant and is independent of p. Hence the ratio of L.H.S. must also be a constant and independent of p. What values must be given to A_p, A_{p-1} and A_{p+1} so that the above condition is satisfied and at the same time

$A_0 = 0$ and $A_{N+1} = 0$.

Let us assume that the amplitude of p[th] particle be represented by $A_p = C \sin p\ q$ … (7)

where q is some angleso that $A_{p-1} = C\sin(p-1)q$

and $A_{p+1} = C\sin(p+1)q$

$$\therefore \quad A_{p-1} + A_{p+1} = C\left[\sin(p-1)q + \sin(p+1)q\right]$$

$$= 2C \sin p\ q \cos q$$

But C sin p q = A_p. $\therefore \frac{A_{p-1} + A_{p+1}}{A_p} = 2_a \cos\theta$

The R.H.S. is independent of p. So that the assumed solution is also found to be true .This will satisfy all N equations. The value of q can be obtained by applying the boundary conditions viz., $A_p = 0$ for p = 0 and p = (N +1). This condition will be satisfied if (N + 1) q is an integral multiple of q i.e.,

(N + 1) q = $n\pi$ Where n = 1, 2, 3, ….

$$\Rightarrow \quad q = \frac{n\pi}{(N+1)} \qquad \ldots (9)$$

Substituting this value of q in Eq. (7), we get

$$A_p = C \sin\frac{pn\pi}{(N+1)} \qquad \ldots (10)$$

The permitted frequencies of the normal modes can be determined from Eq. (6) and (8) as given below. We have

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2} = 2\cos\theta$$

$$\Rightarrow \quad 2\omega_0^2 - \omega^2 = 2\omega_0^2 \cos\theta$$

$$\Rightarrow \quad \omega^2 = 2\omega_0^2(1 - \cos\theta)$$

$$= 2\omega_0^2 \times 2\sin^2\left(\frac{\theta}{2}\right) = 4\omega_0^2 \sin^2\left(\frac{\theta}{2}\right)$$

$$\omega = 2\omega_0 \sin\left(\frac{\theta}{2}\right) \text{ and } \theta = \frac{n\pi}{(N+1)} \quad \ldots (11)$$

In the above Eq. (11), the different values of n give the different normal mode frequencies. Hence in general, Eq. (11), can be written as

$$\omega_n = 2\omega_0 \sin\left[\frac{n\pi}{2(N+1)}\right] \quad \ldots (12)$$

At each frequency, the p^th particle has the amplitude give

by $A_p = C \sin\left[\frac{pn\pi}{(N+1)}\right]$ … (13)

10. *Derive wave equation in a continuous medium. (OR) Derive classical wave equation.*

Ans: The wave equation : The equation of motion of the p^{th} particle is given by.

$$\frac{d^2 y_p}{dt^2} = \frac{T}{ml}\left(y_{p+1} + y_{p-1} - 2y_p\right)$$

Consider the limiting case when $1 = \delta x$ and $\delta x \to 0$

Now the masses merge into a continuous heavy string. In which case

$$\frac{d^2 y_p}{dt^2} = \frac{T}{m}\left[\frac{y_{p+1} + y_{p-1} - 2y_p}{\delta x}\right]$$

$$= \frac{T}{m}\left[\left(\frac{y_{p+1} - y_p}{\delta x}\right) - \left(\frac{y_p - y_{p-1}}{\delta x}\right)\right]$$

But $\left(\frac{dy}{dx}\right)_{x+dx} - \left(\frac{dy}{dx}\right)_x = \left(\frac{d^2 y}{dx^2}\right)dx$

When, the subscripts can be dropped and the equation of motion for the harmonic oscillator at position x can be expressed as $\frac{d^2 y}{dt^2} = \frac{T}{m}\frac{d^2 y}{dx^2}dx = \frac{T}{\rho}\frac{d^2 y}{dx^2}$ … (1)

Since $p = \frac{m}{dx}$ represents the mass per unit length i.e., linear density of the string. Hence Eq. (1) can be written as

$$\frac{d^2 y}{dt^2} = V^2 \frac{d^2 y}{dx^2} \quad \ldots (2)$$

where $v = \sqrt{T/\rho}$ has the dimensions of velocity. Eq. (2) is the second order linear partial differential wave equation. It is called the classical wave equation.

SHORT ANSWER QUESTIONS

11. *Define Simple Harmonic Motion. Give its Characteristics.*

Ans: **Simple Harmonic Motion:** This is a special type of periodic motion in which the body moves along a straight line such that its acceleration is always directed towards a fixed point, and is directly proportional to its displacement but opposite in direction.

Characteristics of simple harmonic motion:

e) The motion is periodic.
f) The motion is along a straight line about the mean position.
g) The acceleration is proportional to displacement and in opposite direction.

h) Acceleration is always directed towards the mean position.

12. Define damping. Explain over damped, critically damped, under damped motions.

Ans: Damping: For an ideal harmonic oscillator, the amplitude of vibration remains constant. When a body vibrates in a medium which offers resistance to its motion, the amplitude of vibration decreases gradually and finally the body comes to rest. This is due to the resistance offered by the medium. Then the motion of the body is known as damped harmonic motion. This phenomenon is called damping

Example: If we displace a pendulum from its equilibrium position it will oscillate with decreasing amplitude and finally comes to rest in equilibrium position.

Over damped motion: In this case the body once displaced returns to its equilibrium position very slowly without performing any oscillation. This type of motion is called as over-damped or dead beat.

Ex: This type of motion is shown by a pendulum moving in thick oil or by a dead beat moving coil galvanometer

Critical Damping: In this case the particle tends to acquire its position of equilibrium much rapidly than inOver damped motion.Such a motion is called critical damped motion.

Ex: Motion of pointer in the instruments such as voltmeter, ammeter etc., In this the pointer moves to correct position and comes to rest without any oscillation in minimum time.

Under damped motion: In thisthe amplitude of the motion is continuously decreasing owing to the factor e^{-bt} which is called damping factor. The amplitude varies between ae^{-bt}

and $-ae^{-bt}$. The decay of the amplitude depends upon the damping coefficient 'b'. It is called "under damped" motion.

Ex: The motion of a pendulum in air, the motion of the coil of ballistic galvanometer or the electric oscillations of LCR circuit.

13. Explain Logarithmic decrement and Relaxation time

Ans: Logarithmic decrement measures the rate at which the amplitude decreases. The amplitude of damped harmonic oscillator is given by, amplitude $= ae^{-bt}$.

At t=0, amplitude $a_0 = a$

Let $a_1, a_2, a_3 ....$ be the amplitudes at time t = T, 2T, 3T …respectively, where T= period of oscillation. Then, $a_1 = ae^{-bT}, a_2 = ae^{-b(2T)}, a_3 = ae^{-b(3T)}...$

From these equations we get, $\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = ... = e^{bT}$

$= e^{\lambda} (where\ \ bT = \lambda)$

Taking the natural logarithm, we get,

$$\log_e e^{\lambda} = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} ....$$

$$\Rightarrow \lambda = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} .....$$

Where λ is known as logarithmic decrement.

Thus logarithmic decrement is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are separated by one period.

Relaxation time: The relaxation time is defined as the time taken for the total mechanical energy to decay to $(1/e)^{th}$ of its original value.

The mechanical energy of damped harmonic oscillator is giveby. $E = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} m\omega^2 a^2 e^{-2bt} \left(\because A = a e^{-bt}\right)$

Let $E = E_0$ *when* $t = 0$,

$$\therefore E_0 = \frac{1}{2} m\omega^2 a^2 \quad Now\, E = E_0 e^{-2bt} \quad \ldots (1)$$

Let τ be the relaxation time, i.e. at t=τ, $E = \frac{E_0}{e}$

$$\frac{E_0}{e} = E_0 e^{-2b\tau} \Rightarrow e^{-1} = e^{-2b\tau} \Rightarrow 2b\tau = 1 \Rightarrow \tau = \frac{1}{2b} \quad \ldots (2)$$

From eqs. (1) and (2) we get $E = E_0 e^{\frac{-t}{\tau}}$ … (3)

The expression of power dissipation can be written as

$$P = \frac{E}{\tau} \quad \ldots (4)$$

14. Explain Quality factor or Q- factor

The quality factor is defined as 2π times the ratio of the energy stored in the system and the energy lost per period.

$$Q = 2\pi \frac{energy\,stored\,in\,system}{energy\,lost\,per\,period} = 2\pi \frac{E}{PT} \quad \ldots (5)$$

Where P is the power dissipatedand T is periodic time. We know that $P = \frac{E}{\tau}$

Where τ is the relaxation time . So,

$$Q = 2\pi \frac{E}{\left(\frac{E}{\tau}\right)T} = \frac{2\pi\tau}{T} \quad \therefore Q = \omega\tau \quad \ldots (6)$$

where $\omega = \frac{2\pi}{T}$ = angular frequency.

15. Define natural and forced vibrations and resonance.

Sol: Natural or free vibrations: When a body is made to vibrate and left free itself it always vibrates with a frequency known as natural frequency. Those vibrations are called natural or free vibrations.

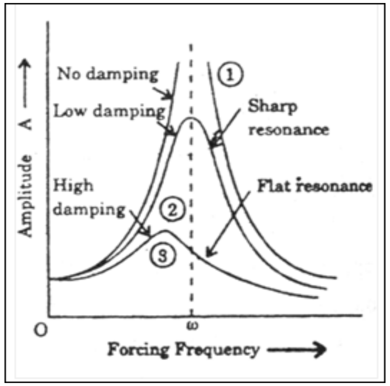
Forced vibrations: When the body vibrates with a frequency other than its natural frequency under the action of an external periodic force, then those vibrations are called Forced vibrations

Amplitude resonance: The amplitude of forced oscillations varies with the frequency of applied force and becomes maximum at a particular frequency. This phenomenon is known as amplitude resonance

16. Explain the Sharpness of resonance.

Sol: We have seen that amplitude of the forced oscillation is maximum when the frequency of the applied force has a value to satisfy the condition of resonance i.e. $p = \sqrt{(\omega^2 - 2b^2)}$. If the frequency changes from this value, the amplitude falls. When the fall in amplitude is very large, for a small change in the resonance condition the resonance is said to be sharp. If the fall in amplitude is small, the resonance is termed as flat.

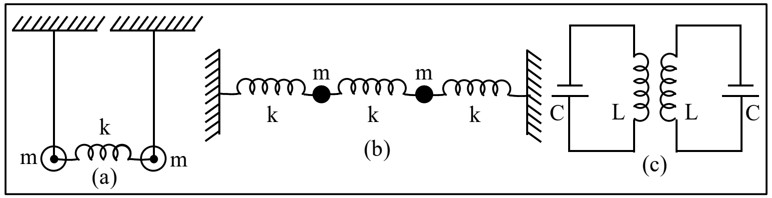
Thus the term sharpness of resonance means the rate of fall in amplitude with the change of forcing frequency on each side of resonance frequency.



The graph shows the variation of amplitude with forcing frequency at different amounts of damping. It is observed from the figure that, the resonance is sharp for smaller damping (curve-2) and the resonance is flat when damping is large(curve-3). Hence, smaller is damping, sharper is resonance or larger is the damping, flatter is the resonance. When there is no damping (curve-1) the amplitude becomes infinity.

17. What are Coupled oscillators. Give examples.

Ans: Coupled oscillators :When the oscillations ofthe oscillators are coupled with one another, they are called coupled oscillators. In this the motion of one oscillator is effected by the other.



Ex: Two coupled oscillating systems are shown in Fig. The fig (a) shows two simple pendulums with their bobs

connected to each other by a spring. Fig (b) shows two masses attached to each other by three springs. The middle spring provides the coupling. Fig (c) shows the coupled LC circuits.

18. What are Normal coordinates and Normal modes

Ans: The *normal coordinate* of a coupled system are the parameters in terms of which the equations of motion of the system can be expressed as a set of linear differential equations with constant coefficients in which each equation contains only one dependent variable.

The S.H.M. associated with each normal coordinate is called a *normal mode* of the coupled system.

A two coupled oscillator has two normal modes, as in phase mode and an out of phase mode. The in phase mode has a frequency equal to the natural frequency of either of the oscillators. The out of phase mode has a frequency slightly greater than the natural frequency of the oscillators.

Significance: the normal modes of vibration are entirely independent of each other. The energy associated with the normal mode is never exchanged with another mode. So we can add the energies of the separate modes to get total energy.

The general motion of any coupled system can always be represented as a superposition of all possible normal modes.

Thus during this motion, the spring is extended and also compressed. Hence the coupling is effective in this case.

19. Derive wave equation in a continuous medium. (OR) Derive classical wave equation.

Ans: The wave equation : The equation of motion of the p[th] particle is given by.

$$\frac{d^2 y_p}{dt^2} = \frac{T}{ml}\left(y_{p+1} + y_{p-1} - 2y_p\right)$$

Consider the limiting case when $1 = \delta x$ and $\delta x \to 0$

Now the masses merge into a continuous heavy string. In which case

$$\frac{d^2 y_p}{dt^2} = \frac{T}{m}\left[\frac{y_{p+1} + y_{p-1} - 2y_p}{\delta x}\right]$$

$$= \frac{T}{m}\left[\left(\frac{y_{p+1} - y_p}{\delta x}\right) - \left(\frac{y_p - y_{p-1}}{\delta x}\right)\right]$$

But $\left(\frac{dy}{dx}\right)_{x+dx} - \left(\frac{dy}{dx}\right)_x = \left(\frac{d^2 y}{dx^2}\right)dx$

When, the subscripts can be dropped and the equation of motion for the harmonic oscillator at position x can be expressed as $\frac{d^2 y}{dt^2} = \frac{T}{m}\frac{d^2 y}{dx^2}dx = \frac{T}{\rho}\frac{d^2 y}{dx^2}$ … (1)

Since $p = \frac{m}{dx}$ represents the mass per unit length i.e., linear density of the string. Hence Eq. (1) can be written as

$$\frac{d^2 y}{dt^2} = V^2 \frac{d^2 y}{dx^2} \qquad \dots (2)$$

where $v = \sqrt{T/\rho}$ has the dimensions of velocity. Eq. (2) is the second order linear partial differential wave equation. It is called the classical wave equation.

SOLVED PROBLEMS

20. A simple harmonics wave is represented by y = 10 sin $\left(\frac{2\pi t}{T} + \theta\right)$. The time period is 30 seconds.

Sol: When t=0, the displacement is 5 cm. Find (a) the phase angle at t=7.5 sec (b) phase difference between two points at a time interval of 6 sec.

Given $y = 10\sin\left(\frac{2\pi t}{T} + \theta\right)$ and T=30 sec when t= 0, y = 5cm = 0.05 m

$$\therefore 5 = 10\sin\theta \Rightarrow \sin\theta = \frac{5}{10} = \frac{1}{2} \Rightarrow \text{phase angle} = 30^0$$

When the time is t_1, y = 10 sin$\left(\frac{2\pi t_1}{T} + \theta\right)$.

When the time is $t_2, y = 10\sin\left(\frac{2\pi t_2}{T} + \theta\right)$

So the phase difference for 6 sec is

$$\phi = \left(\frac{2\pi t_2}{T}+\theta\right)-\left(\frac{2\pi t_1}{T}+\theta\right) = \frac{2\pi}{T}(t_2-t_1)$$

$$=\frac{2\pi}{30}(6)=\frac{2\pi}{5}=72^0$$

21. The displacement equation of a particle describing SHM is x = 0.5 $\cos\left(10pt+\frac{p}{3}\right)$. Calculate (a) amplitude, (b) frequency, (c) phase, (d) displacement after 1 sec.

Sol: Displacement $x = 0.5\cos\left(10\pi t+\frac{\pi}{3}\right)$

we have $x = a\sin(\omega t+\phi)$ Comparing,

a) amplitude $a = 0.5\ m,\ \omega = 2\pi n = 10\pi \Rightarrow$
b) frequency n = 5 Hz
c) Phase $\phi = \frac{\pi}{3}$
d) after 1 sec $x = 0.5\cos\left(10\pi+\frac{\pi}{3}\right) = 0.5\cos\frac{31\pi}{3}$

$= (0.5)(0.5) = 0.25\,m$

22. The displacement of a linear harmonic oscillator is given by x = 4 $\sin\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)$. Find the period and velocity at t = 1 sec.

Sol: Displacement $x = 4\sin\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)$

we have $x = a\sin(\omega t+\phi)$ comparing, $\omega = \frac{\pi}{3}$

$\omega = \frac{2\pi}{T} = \frac{\pi}{3} \Rightarrow T = 6\sec$ *we have* $V = \frac{dx}{dt} = 4\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}t+\frac{\pi}{6}\right)$

23. A simple harmonic wave is represented by x = 10 $\sin\left(\frac{2\pi}{T}t+\theta\right)$. The time period is 30 sec. When t = 0, the displacement is 5cm. Find the phase at t = 7.5 sec. and the phase difference between two points at a time interval of 6 sec.

Sol: phase $\phi = \frac{2\pi}{T}t+\theta = \frac{2\pi(7.5)}{30}+30 = 120^0$

Let ϕ_1 and ϕ_2 be the phase angles corresponding to the times t_1 and t_2. Then,

$$\phi_1 = \frac{2\pi}{T}t_1+\theta \text{ and } \phi_2 = \frac{2\pi}{T}t_2+\theta$$

Phase difference $\phi_2-\phi_1 = \frac{2\pi}{T}(t_2-t_1) = \frac{2\pi}{30}6 = \frac{2\pi}{5}\,rad$

24. A particle executing simple harmonic motion is represented by x = 10 $\sin\left(10t-\frac{\pi}{6}\right)$. Calculate the frequency, time period, maximum displacement, maximum velocity, displacement, velocity and acceleration at time t = 0.

Sol: We know that $x = a\sin(\omega t+\phi)$.

Given $x = 10\sin\left(10t-\frac{\pi}{6}\right)$.

Comparing the two, a=10m, and $\omega = 10\,Hz$

Frequency $n = \dfrac{\omega}{2\pi} = \dfrac{10}{2\pi} = 1.6\,Hz$

Time period $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{10} = 0.63\sec$.

Maximum displacement a = 10 m

Velocity $\dfrac{dx}{dt} = a\,\omega\cos(\omega t + \phi)$.

Velocity is maximum when $\cos(\omega t + \phi) = 1$

$$\therefore \left(\frac{dx}{dt}\right)_{max} = a\,\omega = 10 \times 10 = 100\,ms^{-1}$$

Acceleration $\dfrac{d^2x}{dt^2} = -a\,\omega^2 \sin(\omega t + \phi)$.

Acceleration is maximum when $\sin(\omega t + \phi) = 1$

$$\left(\frac{d^2x}{dt^2}\right)_{max} = -a\,\omega^2 = -10(10)^2 = -1000\,ms^{-2}$$

At t=0, $x = 10\sin\left(-\dfrac{\pi}{6}\right) = -10\left(\dfrac{1}{2}\right) = -5m$

$$\frac{dx}{dt} = 10(10)\cos\left(-\frac{\pi}{6}\right) = 100(0.866) = 86.6\,ms^{-1}$$

$$\left(\frac{d^2x}{dt^2}\right) = -a\,\omega^2 = -10(10)^2 \sin\left(-\frac{\pi}{6}\right) = 1000\left(\frac{1}{2}\right) = 500\ ms^{-2}$$

25. The amplitude of a seconds pendulum falls to half initial value in 150 sec. calculate Q-factor.

Sol: We have $a = a_0\, e^{-bt} \Rightarrow \dfrac{a}{a_0} = e^{-bt}$ given $\dfrac{a}{a_0} = \dfrac{1}{2}$

and t = 150 sec.

$$\therefore \frac{1}{2} = e^{-b\,(150)} \Rightarrow e^{b\,(150)} = 2 \Rightarrow 150b = \log_e(2) = 2.303\,\log_{10}(2)$$

$$\Rightarrow 150b = 0.6932 \Rightarrow b = \frac{0.6932}{150} = 0.00462$$

we have $\tau = \dfrac{1}{2b} = \dfrac{1}{0.00924}$ and $\omega = \dfrac{2\pi}{T}$

$$= \frac{2(3.14)}{2} = 3.14\,rad\,\sec^{-1}$$

$Q-$factor $Q = \tau\,\omega = \left(\dfrac{1}{0.00924}\right) 3.14 \approx 340$

26. The amplitude of an oscillator of frequency falls to 1/10 th of its initial value after 2000 cycles. Calculate i) relaxation time ii) Q- factor iii) time in which its energy falls to 1/10th of initial value iv) damping constant.

Sol: We have $a = a_0\, e^{-bt} \Rightarrow \dfrac{a}{a_0} = e^{-bt}$

Given $\dfrac{a}{a_0} = \dfrac{1}{10}$ and t = 10 sec.

$$\therefore \frac{1}{10} = e^{-b\,(10)} \Rightarrow e^{b\,(10)} = 10$$

$$\Rightarrow 10b = \log_e(10) = 2.303\ \log_{10}(10)$$

$$\Rightarrow\ 10b = 2.303 \Rightarrow b = \frac{2.303}{10} = 0.2303$$

We have relaxation time $\tau = \frac{1}{2b} = \frac{1}{2(0.2303)} = 2.174\,\text{sec}$

$$\therefore\ Q - factor\ Q = \tau\ \omega = \tau(2\pi n)$$

27. *The Q-value of a spring loaded with 0.3 kg is 60. It vibrates with a frequency of 2 Hz. Calculate the force constant and mechanical resistance.*

Sol: frequency $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ where k = force constant

$$\therefore\ 2 = \frac{1}{2\pi}\sqrt{\frac{k}{0.3}} \Rightarrow 4 = \frac{1}{4\pi^2}\frac{k}{0.3}$$

$$\Rightarrow k = 16(\pi^2)0.3 = 47.37\ Nm^{-1}$$

We have $\tau = \frac{1}{2b} = \frac{1}{\left(\frac{r}{m}\right)} = \frac{m}{r}$ *and* $\omega = 2\pi n$

$$\therefore\ Q - factor\ \ Q = \tau\ \omega = \frac{m}{r}(2\pi n)$$

$$\Rightarrow\ r = \frac{(2\pi n)m}{Q} = \frac{2(0.34)2(0.3)}{60} = 0.6282\,kg\,s^{-1}$$

28. *A condenser of capacity 20 μF is discharged through an inductance of 10 mH. Calculate the frequency of resultant oscillation.*

Sol: frequency $n = \frac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{(10\times10^{-3})(20\times10^{-6})}} = 356.12\,Hz$$

29. *Deduce the frequency and Q – factor for a circuit with L = 2 mH, C = 5 μF and R = 0.2 Ω.*

Sol: frequency $n = \frac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{(2\times10^{-3})(5\times10^{-6})}} = 1592\,Hz$$

$$Q = \frac{\omega L}{R} = \frac{(2\pi n)L}{R} = \frac{2(3.14)1592(2\times10^{-3})}{0.2} = 99.9$$

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UNIT-V

7. VIBRATING STRINGS

ESSAY QUESTIONS

1. *Derive the general wave equation for a transverse wave propagation along a stretched string and its general solution.*

Ans: **:** Consider a string stretched in positive X- direction in which a teraservers wave travelling. Consider a point P at a distance x from the origin O . The vibrations at P lags behind a phase ϕ. Let V be the velocity of the wave in positive X- direction.

The equation of motion of a particle at O is $y = a\sin\omega t$ So the equation of motion of a particle at O is $y = a\sin(\omega t - \phi)$

But $phase\ difference = \frac{2\pi}{\lambda}(path\ \ difference)$

$$\Rightarrow \phi = \frac{2\pi}{\lambda}x$$

$$\therefore y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right) \Rightarrow y = a\sin\left(2\pi n t - \frac{2\pi}{\lambda}x\right)$$

$$\Rightarrow y = a\sin\frac{2\pi}{\lambda}(\lambda n t - x) \Rightarrow y = a\sin\frac{2\pi}{\lambda}(Vt - x)$$

If we consider the propagation along negative X- direction $y = a\sin\frac{2\pi}{\lambda}(V t + x)$

$$\therefore y = f\ (Vt \pm x) \qquad \ldots (1)$$

Differentiating the equation (1) w.r.t.timet,

$$\therefore \frac{dy}{dt} = V f^{1}(Vt \pm x)$$

$$\Rightarrow \frac{d^2 y}{dt^2} = V^2 f^{||}(Vt \pm x) \ldots \qquad (2)$$

Differentiating the equation (1) w.r.t. x,

$$\frac{dy}{dx} = \pm f^{1}(Vt \pm x) \Rightarrow \frac{d^2 y}{dt^2} = f^{||}(Vt \pm x) \qquad \ldots (4)$$

$$\therefore (2) \Rightarrow \frac{d^2 y}{dt^2} = V^2 \frac{d^2 y}{dx^2}$$

This is the general wave equation.

General solution of wave equation: the general solution can be written as,

$$y = f_1\ (Vt + x) + f_2\ (Vt - x)$$

In simple harmonic terms the solution can be written as,

$$y = a\ \sin(\omega t \pm k x)$$

Where ω the angular frequency and k is the propagation constant.

We have $\omega = \frac{2\pi}{n}$ and $k = \frac{2\pi}{\lambda}$

$$y=a\sin\left(2\pi nt\pm\frac{2\pi}{\lambda}x\right)\Rightarrow y=a\sin\frac{2\pi}{\lambda}(n\lambda t\pm x)$$

$$\Rightarrow y=a\sin\frac{2\pi}{\lambda}(Vt\pm x) \qquad ....(1)$$

This is called the differential form of the wave equation.

General solution of wave equation: Any arbitrary functions of the form $(Vt+x)$ or $(Vt-x)$ will be the solution of the wave equation. Hence the most general solution will be the linear combination of the two.

$$\therefore \quad y=f_1(Vt+x)+f_2(Vt-x) \qquad \ldots(2)$$

In simple harmonic terms the motion of a particle can be expressed by $a\sin(\omega t\pm\phi)$ or $a\cos(\omega t\pm\phi)$. But we have $\phi=\frac{2\pi}{\lambda}x$

$$\therefore \quad y=a\sin\left(\omega t\pm\frac{2\pi}{\lambda}x\right) \text{ or } y=a\cos\left(\omega t\pm\frac{2\pi}{\lambda}x\right)$$

$$\Rightarrow \quad y=a\sin\left(2\pi n\,t\pm\frac{2\pi}{\lambda}x\right)\Rightarrow \quad y=a\sin\frac{2\pi}{\lambda}(n\lambda t\pm x)$$

$$\Rightarrow \quad y=a\sin\frac{2\pi}{\lambda}(vt\pm x)\Rightarrow \quad y=a\sin\left(\omega t\pm\frac{2\pi}{\lambda}x\right)$$

$$\Rightarrow \quad y=a\sin(\omega t\pm kx)\text{where } k=\frac{2\pi}{\lambda}$$

or $\quad y=a\cos(\omega t\pm kx)$

Hence the general simple harmonic solution can be written as,

$$y=a_1\sin(\omega t+kx)+a_2\sin(\omega t-kx)$$
$$+b_1\cos(\omega t+kx)+b_2\cos(\omega t-kx)$$

Significance : consider a transverse wave travels along a medium. By substituting the boundary conditions for the displacement x we can obtain the positions of nodes and antinodes in the medium along the wave propagation.

2. Derive an expression for the velocity of a transverse wave along a stretched string.

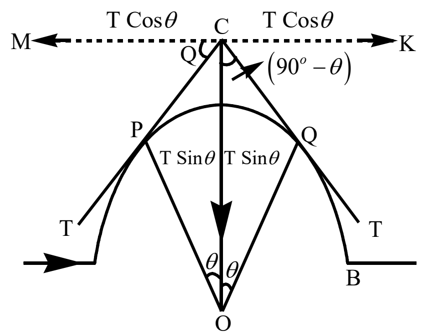
Sol: Consider a perfectly flexible, uniform string with linear density 'm' kept under a constant tension 'T'. Let a transverse wave travel along the string as shown. Consider a small part of the string 'PQ'. It can be considered as an arc of a circle of radius r with center 'O'. The tension T in the string acts along the tangents drawn at the points 'P' and 'Q'. let C be the point of intersection of the two tangents. OC is the bisector of the angle $\angle POQ$.

$$\text{Let}\angle POQ=2\theta \quad \Rightarrow\angle POC=\angle QOC=\theta$$

$$and\ \angle OPC=\angle OQC=90^0$$

$$\therefore\ \angle PCO=\angle QCO=(90-\theta)$$

The tension at P and Q can be resolved into two rectangular components. The horizontal components $T\cos\theta$ along CM and CK cancel each other. The vertical components along CO add together and form the necessary centripetal force.



Component of tension at P along

$$CO = T\cos(90-\theta) = T\sin\theta$$

Component of tension at Q along

$$CO = T\cos(90-\theta) = T\sin\theta$$

Total tensional force acting along CO = $T\sin\theta$ + $T\sin\theta$ = 2 $T\sin\theta$

$= 2\,T\theta \quad (\because \theta \text{ is very small } \sin\theta \approx \theta)$

From the sector PQO, $angle = \dfrac{arc}{radius}$

$$\Rightarrow 2\theta = \frac{PQ}{r} \Rightarrow \theta = \frac{PQ}{2r}$$

Tensional force acting along CO = $2T\theta = 2T\dfrac{PQ}{2r} = \dfrac{T(PQ)}{r}$

Centripetal force = $\dfrac{mass(velocity)^2}{r}$

Mass of the considered element PQ = linear density x length = m (PQ)

$\therefore$ Centripetal force = $\dfrac{m(PQ)V^2}{r}$

$$Hence\ \frac{T(PQ)}{r} = \frac{m(PQ)V^2}{r} \Rightarrow T = mV^2$$

$$V^2 = \frac{T}{m} \Rightarrow V = \sqrt{\frac{T}{m}}$$

3. *Explain the modes of vibration of stretched string clamped at ends.*

Sol: For a stretched string the general wave equation solution can be taken as,

$$x = a_1\sin(\omega t - kx) + a_2\sin(\omega t + kx)$$
$$+ b_1\cos(\omega t - kx) + b_2\cos(\omega t + kx)$$

Where a_1, a_2, b_1, b_2 are constants

As the string is rigidly supported at the ends, the boundary conditions are, *y=0* at *x=0* at all time't' *y=0* at *x=l* at all time*t*.

Applying boundary conditions in eq (1), we get,

$$0 = a_1\sin\omega t + a_2\sin\omega t + b_1\cos\omega t + b_2\cos\omega t$$

$$0 = (a_1 + a_2)\sin\omega t + (b_1 + b_2)\cos\omega t$$

As $\sin\omega t \neq$ o, and $\cos\omega t \neq$ o

Hence, $(a_1 + a_2)$ =0 and $(b_1 + b_2)$=0

Thus we have, $a_1 = -a_2$ and $b_1 = -b_2$

Now eq(1) becomes

$$y = a_1[\sin(\omega t - kx) - \sin(\omega t + kx)]$$
$$+ b_1[\cos(\omega t - kx) - \cos(\omega t + kx)]$$

$$= a_1[\{\sin\omega t\cos kx - \cos\omega t\sin kx\} - \{\sin\omega t\cos kx + \cos\omega t\sin kx\}$$
$$+ b_1[\{\cos\omega t\cos kx + \sin\omega t\sin kx\} - \{\cos\omega t\cos kx - \sin\omega t\sin kx\}]$$

$$= -2a_1\cos\omega t\sin kx + 2b_1\sin\omega t\sin kx$$

$$= (-2a_1\cos\omega t + 2b_1\sin\omega t)\ \sin kx$$

The solution now consists of two terms, one depending on *'t'* and second on *'x'*. Thus the first boundary condition reduces the opposite waves to a stationary wave. Now we apply the boundary condition to eq. (3). As $\sin\omega t \neq$ o, and $\cos\omega t \neq$ o, hence $\sin kl = 0 \ \Rightarrow kl = n\pi$

where n=1,2,3,……

$\Rightarrow k_n = \frac{n\pi}{l}$, where n=1,2,3,… $\Rightarrow v_n = n\left(\frac{v}{2l}\right)$

where n=1,2,3….

$$\left[\because k = \frac{2\pi}{\lambda} = \frac{2\pi v}{V} \Rightarrow v = \frac{kV}{2\pi} \qquad v = \frac{n\pi V}{2\pi l} = n\left(\frac{V}{2l}\right)\right]$$

This equation represents the mode of vibration corresponding to n^{th} harmonic frequency. The different modes of vibration are shown in fig .

The fundamental frequency corresponding to n=1 is given by,

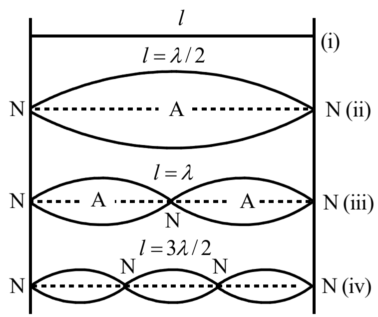
$$v_1 = \frac{V}{2l} = \frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)} \qquad \left[\because V = \sqrt{\left(\frac{T}{m}\right)}\right]$$

The n^{th} harmonic mode of frequency is given by

$$v_n = \frac{n}{2l}\sqrt{\left(\frac{T}{m}\right)}$$

OVERTONES AND HARMONICS:

Let us consider the case of a string fixed at the two ends and plucked [fig.].the progressive wave generated in the string travel to the both ends. As the two ends are fixed, no displacement is possible at these two ends and the waves are reflected with phase change. The two waves interfere and produce stationary wave with nodes at fixed ends. The frequency of vibration will depend on the number of nodes formed between two fixed ends of the string.



When the string is plucked at the middle, it vibrates with nodes at the ends and antinodes at the middle[fig]. The tone emitted under this condition is known as *fundamental or first harmonic* .the frequency v_1 is given by

$$l = \frac{\lambda}{2} \Rightarrow \lambda = 2l \text{ we have } v = \frac{1}{\lambda}\sqrt{\left(\frac{T}{m}\right)} \Rightarrow v_1 = \frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$$

If the string is plucked at the one-fourth of its length, the string vibrates in two segments. In this case there is one more node[fig.iii] at the middle point. The frequency of the vibration of the string is given by ,

$$l = \lambda \quad \text{we have } v = \frac{1}{\lambda}\sqrt{\left(\frac{T}{m}\right)} \quad \therefore v = \frac{1}{l}\sqrt{\left(\frac{T}{m}\right)}$$

$$v_2 = \frac{2}{2l}\sqrt{\left(\frac{T}{m}\right)} = 2v_1$$

This is called first overtone or second harmonic.

When the string vibrates in three segments [fig (iv)], the frequency of the vibration is given by

$$l = \frac{3\lambda}{2} \quad \text{we have } v = \frac{1}{\lambda}\sqrt{\left(\frac{T}{m}\right)} \qquad \therefore v_3 = \frac{3}{2l}\sqrt{\left(\frac{T}{m}\right)} = 3v_1$$

This is called second overtone or third harmonic.

Similarly, when the string vibrates in four segments, then

$$v_4 = \frac{4}{2l}\sqrt{\left(\frac{T}{m}\right)} = 4v_1$$

This is called third overtone or fourth harmonic.

So in case of stretched string, we have

$v_1 : v_2 : v_3 ..... = 1 : 2 : 3....$

Thus the string fixed at both ends has all possible harmonics.The frequencies of the harmonics are integral multiple of the fundamental.

4. Explain about Melde's strings.

Sol: The strings used in Melde's experiment for the demonstration of transverse wave velocity along a stretched string are called Melde's strings.

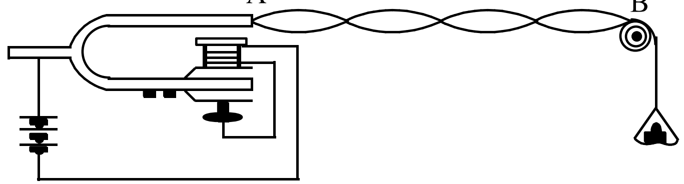
A thread of uniform radius and density is attached to one of the prongs of an electrically maintained tuning fork and the other end passes over a frictionless pulley and carries a known weight. As the fork vibrates, plane progressive waves travel along the thread and is reflected back from the pulley. By properly adjusting the length and the tension, the thread can be made into stationary wave pattern with well defined nodes. There are two possible arrangements for performing the experiment,

(1) transverse arrangement and (2) longitudinal arrangement.

(1) Transverse arrangement: In this arrangement, the fork is placed so that the motion of the prongs is at right angle to the thread as shown in Fig. Stationary waves are produced due to the superposition of direct waves sent by tuning fork and reflected waves from pulley. It can be shown that as the fork completes its one vibration, the thread also completes one vibration. Thus the frequency of the thread is the same as that of the fork. If the thread vibrates in one loop the frequency of fundamental node is given by

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}} \qquad \ldots (1)$$

Where l is its length, T the tension and m the mass per unit length.



Here m and n are fixed, hence the vibrations of the thread are maintained by adjusting the length of the thread or the tension or both. If the same length vibrates in p loops under a tension T_p, then

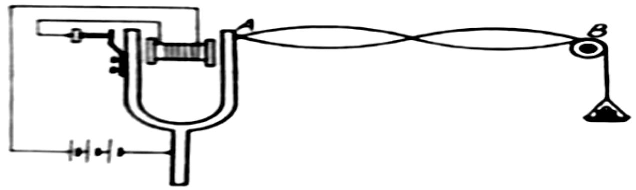
$$n = \frac{p}{2l}\sqrt{\frac{T_p}{m}} \qquad \ldots (2)$$

From equations (1) and (2), we have

$$\frac{T_p}{T} = \frac{1}{p^2} \text{ or } T_p = \frac{T}{p^2} \qquad \ldots (3)$$

This equation shows that if we wish to have p loops in the same length then the tension should be reduced to $1/p^2$ of its previous value.

(2) Longitudinal arrangement : In this arrangement the fork is placed in such a way that the motion of the prongs is along the length of the thread as shown in Fig.



In this case the frequency of the thread is one half of that of the fork ,because in two complete vibrations of the tuning fork, the thread completes one vibration. Thus the frequency of the thread is one-half of the fork i.e., n/2.

If the same length l of the thread vibrates in one loop under a tension T^1, then

$$\frac{n}{2}=\frac{1}{2l}\sqrt{\frac{T^1}{m}} \qquad \ldots (4)$$

Let T_p^1 be the tension applied when the thread vibrates in p lops, then

$$\frac{n}{2}=\frac{p}{2l}\sqrt{\left(\frac{T_p^1}{m}\right)} \qquad \ldots (5)$$

From equations (4) and (5), we get,

$$\frac{T_p^1}{T^1}=\frac{1}{p^2} \qquad \ldots (6)$$

Again comparing equations (2) and (5), we have

$$T_p^1=\frac{T_p}{4} \qquad \ldots (7)$$

Hence in the longitudinal arrangement the same length of the thread under the same tension will vibrate in half the number of loops than in the transverse arrangement.

5. Mention the laws of transverse vibrations.

Sol: Wehave frequency $v_1=\frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$. Hence

(1) The frequency of the fundamental note emitted is inversely proportional to the length of the string when tension and mass per unit length are constant, i.e..

$$v \ \alpha \ \frac{1}{l} \text{ when T and m are constants}$$

(2) The frequency of the fundamental note emitted is directly proportional to the square root of the tension when the length and linear density are constants. i.e..

$$v \ \alpha \ \sqrt{T} \text{ when } l \text{ and m are constants.}$$

(3) The frequency of the fundamental note emitted is inversely proportional to the square root of mass per unit length of the wire when length and tension are constant i.e..

$$v \ \alpha \ \frac{1}{\sqrt{m}} \text{ when } l \text{ and t are constants}$$

8. ULTRASONICS

6. What are Ultrasonic waves? Mention the general Properties of ultrasonic waves.

Ans: Sound waves having frequencies from 20Hz to 20,000Hz are called audible sounds.

Sound waves having frequencies less than 20Hz are called infrasonic waves.

Sound waves having frequencies greater than 20,000Hz are called Ultrasonic waves.

Properties of ultrasonic waves:

1. These are highly energetic. Their speed of propagation depends on their frequency.
2. They have small amount of diffraction due to their smaller wavelength.
3. Due to their smaller wavelength, ultrasonic can be transmitted over long distances without much loss in energy.

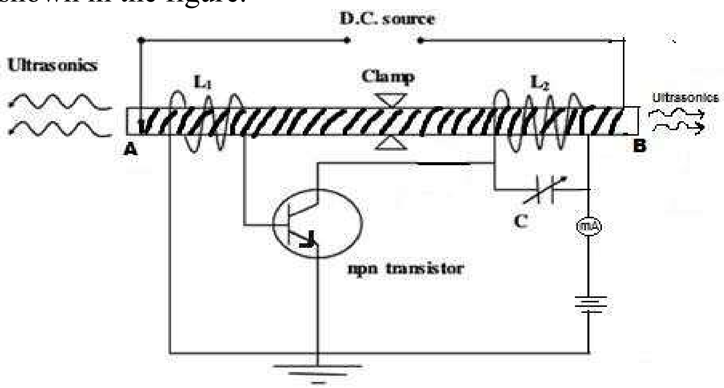
4. Intensive ultrasonic waves have a destructive effect on liquids, by causing bubbles in them.
5. When a plane stationary ultrasonic wave is produced in a liquid, it acts as a grating todiffract the light.

7. *Explain the production of ultrasonic waves by magnetostriction method.*

Ans: Definition: A rod of ferromagnetic material undergoes changes in its length when an alternating magnetic field is applied parallel to its length. This phenomenon is called magneto striction.

The change in length depends only on the nature of the material and the magnitude of the field.

Experimental arrangement: The experimental arrangement is shown in the figure.



It consists of a ferromagnetic rod AB which is clamped at the middle. The rod is permanently magnetized in the beginning by passing a direct current in the coil which is wrapped round the rod. There are two other coils L_1 and L_2 which are wrapped along the rod as shown. The coil L_2 is connected in the collector circuit while L_1 is connected in the

base circuit of a NPN transistor. The frequency of the oscillating collector circuit is adjusted with the help of a variable condenser 'C'. When the frequency of the collector circuit is same as the natural frequency of the rod resonant vibrations are produced. These vibrations are maintained due to coupling provided by the coil L_1.

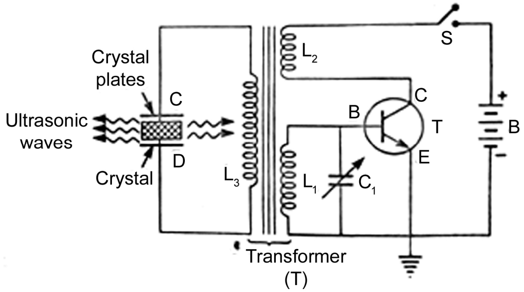
Working: When the collector current passing through the coil L_2 is changed, it causes a corresponding change in the magnetization of the rod. Hence there is a change in the length of the rod. This variation in length causes a variation in the magnetic flux of the coil L_1 and an induced emf is produced. It is connected to the base as feedback. So the collector current changes correspondingly. By this cyclic action, the vibrations of the rod are maintained.

8. *Explain the production of ultrasonic waves by piezoelectric method.*

Ans: Certain crystals like quartz or tourmaline have a peculiar property when such crystal slice is cut properly and pressure is applied to its opposite faces, then equal and opposite charges are developed across the perpendicular faces. This is known as Piezo-electric effect. When the applied pressure is changed to tension, the sign of the charges is reversed.

The converse of piezo-electric effect is also true. When a potential difference is applied on the opposite faces of piezo-electric crystal it undergoes deformation along the perpendicular faces. This converse effect is used in the production of ultrasonics.

Experimental arrangement:The arrangement consists of a quartz crystal PQ, which shows a piezo-electric effect. It is placed between two electric plates C and D. These plates form a condenser with crystal as dielectric. The plates C and D are connected to the secondary coil L_3 of the transformer. L_1 is the base coil of the transistor which is coupled with collector coil L_2.The coil L_1 contains a variable capacitor C_1 . The transistor circuit acts as an oscillator. The frequency of its oscillation is controlled by the variable capacitor.



By the action of transformer an oscillatory emf is induced in the coil L_3.Theemf is applied on the crystal along the plates C and D. The capacity of the condenser is changed until the frequency of tuned circuit matches with the natural frequency of the crystal. Then the crystal vibrates and produce ultrasonic wave.

9. Explain about the methods for detecting the ultrasonic waves.

Ans: ***(1) Kundt's tube method:*** When ultra sonic waves are passed through Kundt's tube the lycopodium powder sprinkled in the tube collects in the form of heaps at the nodes. The average distance between two consecutive heaps gives the value of $\frac{\lambda}{2}$. From that value the wavelength and hence the frequency of the waves can be estimated.

(2) Sensitive Flame method:A narrow sensitive flame is moved along the medium to detect ultrasonic waves. The flame remains steady at antinodes and flickers at nodes. This is due to maximum change in pressure. Then by noting the positions of nodes and antinodes the wave length and the frequency can be estimated.

(3)Thermal detector method:In this method a fine platinum wire is moved in the medium of ultrasonic waves. The temperature of the wire changes, because of alternate compressions and rarefactions. This temperature change occurs at nodes, while the temperature remains constant at antinodes. Hence there is a change in the resistance of platinum wire at nodes and remains constant at antinodes. This change in resistance can be detected by using Whetstone's network. The bridge will be in balanced position at antinodes. From this the wavelength and the frequency can be estimated.

(4) Piezo-electric detector: The quartz crystal can also be used for the detection of ultrasonics. When one pair of faces of quartz crystal is subjected to ultrasonics, opposite charges are developed on the other pair, perpendicular to the first. These charges are very small. So they are amplified and detected by suitable methods.

10. Explain the applications of ultrasonic waves.

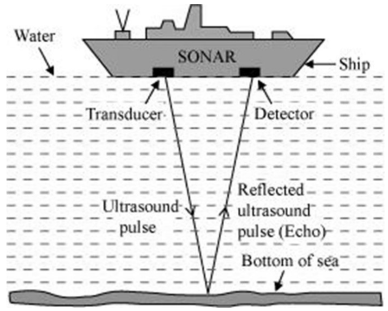
Ans: Because of smaller wavelengths ultrasonics are used in a wide range of regions.

(i) These are used for the detection of structure of matter.
(ii) The flaws in metals can be detected by using these waves.

(iii) These waves can be used for cleaning utensils, washing clothes, removing dust.
(iv) In SONAR these waves are used for detecting the depth of sea.
(v) These are used for signaling in under water communication for the detection of iceberg, submarines under water.
(vi) These are used to produce alloys of uniform alloys of uniform composition.
(vii) These waves can be used for drilling and cutting processes in metals and also forSoldering.
(viii) Physical and chemical effects:
a) These are used to produce emulsions of immiscible like water and oil.
b) These are used to produce colloidal solutions of metals.
c) To coagulate small suspended particles.
d) To release trapped gases in metallurgy.
(ix) Medical and biological effects:
a) Animals like rats, frogs, fishes etc. Can be killed or injured by high intensity ultrasonics.
b) A great relief can be obtained for neurologic pains by exposing those parts to ultrasonics.
c) Abnormal growth in brain, certain tumours can be detected by ultrasonics.
d) Ultrasonics are used in bloodless surgery.

11. Explain about SONAR system.

Ans:**Sonar** is an acronym for **SO**und***N***avigation ***A***nd***R***anging. It is a technique that uses sound propagation to communicate or to detect objects on or under the surface of the water.



Sonar consists of transmitter and a detector and is installed in a boat or a ship as shown in Fig. The transmitter produces and transmits ultrasonic waves. These waves travel through water and after striking the object on the seabed, get reflected back and are sensed by the detector. The detector converts the ultrasonic waves into electrical signals which are appropriately interpreted. The distance of the object that reflected the sound wave can be calculated by knowing the speed of sound in water and the time interval between transmission and reception of the ultrasound.

Let the time interval between transmission and reception of ultrasound signal be t and the speed of sound through sea water be V. The total distance, 2d travelled by the ultrasound is then, $2d = Vt$

Applications : The sonar technique is used to determine the depth of the sea and to locate underwater hills, valleys, submarine, icebergs, sunken ship etc.

SHORT ANSWER QUESTIONS

12. Derive an expression for the velocity of a transverse wave along a stretched string.

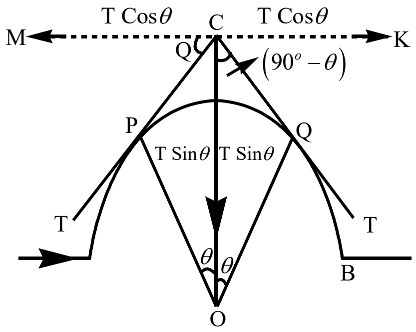
Ans: Consider a perfectly flexible, uniform string with linear density '*m*' kept under a constant tension 'T'. Let a transverse wave travel along the string as shown. Consider a small part of the string 'PQ'. It can be considered as an arc of a circle of radius r with center 'O'. The tension T in the string acts along the tangents drawn at the points 'P' and 'Q'. let C be the point of intersection of the two tangents. OC is the bisector of the angle $\angle POQ$.

Let $\angle POQ = 2\theta \quad \Rightarrow \angle POC = \angle QOC = \theta$

$and\ \angle OPC = \angle OQC = 90^0$

$\therefore \angle PCO = \angle QCO = (90 - \theta)$

The tension at P and Q can be resolved into two rectangular components. The horizontal components $T\cos\theta$ along CM and CK cancel each other. The vertical components along CO add together and form the necessary centripetal force.



Component of tension at P along

$$CO = T\cos(90-\theta) = T\sin\theta$$

Component of tension at Q along

$$CO = T\cos(90-\theta) = T\sin\theta$$

Total tensional force acting along CO = $T\sin\theta$ + $T\sin\theta$ = $2\ T\sin\theta$

$= 2\ T\theta \quad (\because \theta\ is\ very\ small\ \sin\theta \approx \theta)$

From the sector PQO, $angle = \dfrac{arc}{radius}$

$$\Rightarrow 2\theta = \frac{PQ}{r} \Rightarrow \theta = \frac{PQ}{2r}$$

Tensional force acting along CO = $2T\theta = 2T\dfrac{PQ}{2r} = \dfrac{T(PQ)}{r}$

Centripetal force = $\dfrac{mass(velocity)^2}{r}$

Mass of the considered element PQ = linear density x length = m (PQ)

$\therefore$ Centripetal force = $\dfrac{m(PQ)\,V^2}{r}$

$$Hence\ \frac{T(PQ)}{r} = \frac{m(PQ)\,V^2}{r} \Rightarrow T = mV^2$$

$$V^2 = \frac{T}{m} \quad \Rightarrow V = \sqrt{\frac{T}{m}}$$

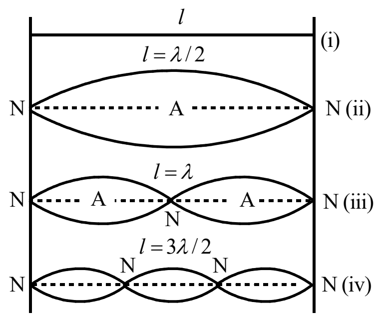
13. Explain the modes of vibration of stretched string clamped at ends.

Ans: The nth harmonic mode of frequency is given by

$$v_n = \frac{n}{2l}\sqrt{\left(\frac{T}{m}\right)}$$

OVERTONES AND HARMONICS:

Let us consider the case of a string fixed at the two ends and plucked [fig.].the progressive wave generated in the string travel to the both ends. As the two ends are fixed, no displacement is possible at these two ends and the waves are reflected with



phase change. The two waves interfere and produce stationary wave with nodes at fixed ends. The frequency of vibration will depend on the number of nodes formed between two fixed ends of the string.

When the string is plucked at the middle, it vibrates with nodes at the ends and antinodes at the middle[fig]. The tone emitted under this condition is known as *fundamental or first harmonic* .the frequency v_1 is given by

$$l=\frac{\lambda}{2} \Rightarrow \lambda=2l \text{ we have } v=\frac{1}{\lambda}\sqrt{\left(\frac{T}{m}\right)} \Rightarrow v_1=\frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$$

If the string is plucked at the one-fourth of its length, the string vibrates in two segments. In this case there is one more node[fig.iii] at the middle point. The frequency of the vibration of the string is given by ,

$$l=\lambda \quad \text{we have } v=\frac{1}{\lambda}\sqrt{\left(\frac{T}{m}\right)} \quad \therefore v=\frac{1}{l}\sqrt{\left(\frac{T}{m}\right)}$$

$$v_2=\frac{2}{2l}\sqrt{\left(\frac{T}{m}\right)}=2v_1$$

This is called first overtone or second harmonic.

When the string vibrates in three segments [fig (iv)], the frequency of the vibration is given by

$$l=\frac{3\lambda}{2} \text{ we have } v=\frac{1}{\lambda}\sqrt{\left(\frac{T}{m}\right)} \qquad \therefore v_3=\frac{3}{2l}\sqrt{\left(\frac{T}{m}\right)}=3v_1$$

This is called second overtone or third harmonic.

Similarly, when the string vibrates in four segments, then

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So in case of stretched string, we have

$$v_1 : v_2 : v_3 ..... = 1 : 2 : 3 ....$$

Thus the string fixed at both ends has all possible harmonics.The frequencies of the harmonics are integral multiple of the fundamental.

14. What are Melde's strings explain.

Ans: The strings used in Melde's experiment for the demonstration of transverse wave velocity along a stretched string are called Melde's strings.

A thread of uniform radius and density is attached to one of the prongs of an electrically maintained tuning fork and the other end passes over a frictionless pulley and carries a known weight. As the fork vibrates, plane progressive waves travel along the thread and is reflected back from the pulley. By properly adjusting the length and the tension, the thread can be made into stationary wave pattern with well defined nodes. There are two possible arrangements for performing the experiment,

(1) transverse arrangement and (2) longitudinal arrangement.

15. Mention the laws of transverse vibrations.

Ans: Wehave frequency $v_1=\frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$. Hence

(3) The frequency of the fundamental note emitted is inversely proportional to the length of the string when tension and mass per unit length are constant, i.e..

$$v \ \alpha \ \frac{1}{l} \text{ when T and m are constants}$$

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(3) The frequency of the fundamental note emitted is inversely proportional to the square root of mass per unit length of the wire when length and tension are constant i.e..

$$v \ \alpha \ \frac{1}{\sqrt{m}}$$ when l and t are constants

16. *What are* audible sounds,infrasonic and *Ultrasonic waves?*

Ans: Sound waves having frequencies from 20Hz to 20,000Hz are called audible sounds.

Sound waves having frequencies less than 20Hz are called infrasonic waves.

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17. Give the Properties of ultrasonic waves

1. These are highly energetic. Their speed of propagation depends on their frequency.
2. They have small amount of diffraction due to their smaller wavelength.
3. Due to their smaller wavelength, ultrasonic can be transmitted over long distances without much loss in energy.

4. Intensive ultrasonic waves have a destructive effect on liquids, by causing bubbles in them.
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18. Definemagnetostrictionand piezo-electric effect

Ans: Magnetostriction: A rod of ferromagnetic material undergoes changes in its length when an alternating magnetic field is applied parallel to its length. This phenomenon is called magnetostriction.

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iii) These waves can be used for cleaning utensils, washing clothes, removing dust.

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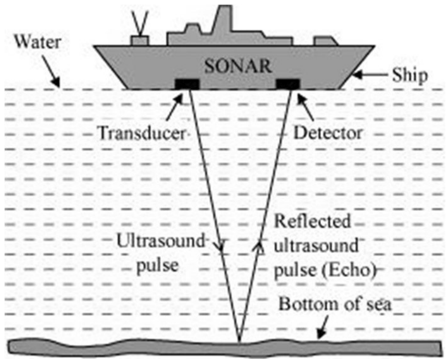


Let the time interval between transmission and reception of ultrasound signal be t and the speed of sound through sea water be V. The total distance, 2d travelled by the ultrasound is then, $2d = Vt$

Applications : The sonar technique is used to determine the depth of the sea and to locate underwater hills, valleys, submarine, icebergs, sunken ship etc.

SOLVED PROBLEMS

22. The progressive wave along a string has a frequency 40 Hz and a wavelength 50 cm. The amplitude is 5 mm. From the given data, write the wave equation in S.I. system.

Sol: ***l:*** Frequency n = 40 Hz

But $\omega = 2\pi\ n$ = 80 rad/sec.

Wavelength $\lambda = 50$ cm = 0.5 m

Wave number $k = 2\pi/\lambda = 2\pi/0.5 = 4\pi$

and amplitude a = 5 mm = 0.005m

The wave equation is represented by y = a sin (kx - ωt)

$\Rightarrow y = 0.005\sin\left(4\pi x - 80\pi t\right)$

23. A steel wire of diameter 1 cm is kept under a tension of 5Kn. The density of steel is 7.8 gm/cm^2. Calculate the velocity of waves.

Sol: The speed of transverse wave v in a string of linear density m under a tension T is given by

$v = \sqrt{T/m}$

Here T = 5 KN = 5000 N;

m = cross section × density =

$$\pi\left(1/2\right)\left(10^{-2}\right)^2 \times \left[7.8\times10^3\right] kg/m$$

$$\therefore v = \sqrt{\frac{5\times1000}{\pi\left(0.5\right)\left(10^{-2}\right)^2\times\left[7.8\times10^3\right]}} = 12.26 m/\text{sec}.$$

24. A string of length 8 m fixed at both ends has a tension of 49 N and a mass 0.04 kg. Find the speed of transverse waves on this string.

Sol: Here $m = \frac{0.04}{8} = 0.005 kg/m$; T = 49 N and v = ?

But $v = \sqrt{\frac{T}{m}} = \sqrt{\frac{49}{0.005}} = 98 m/\sec$

25. The velocity of a wave in a stretched string of tension 19.6 N is 500 m/s. Find the velocity of a wave in that string with a tension of 78.4N.

Sol: The speed of a transverse wave v in a string of linear density m under a tension T is given by

$$v = \sqrt{T/m} \quad \Rightarrow v \propto \sqrt{T} \quad \Rightarrow \frac{V_1}{V_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\frac{v}{500} = \sqrt{\frac{78.4}{19.6}} \Rightarrow v = 500\sqrt{4} = 500 \times 2 = 1000 ms^{-1}$$

26. Calculate the speed of transverse wave in a wire of 1 mm² cross-section under the tension produced by 0.1 kg. wt. The density of the material of the wire is 9.81 × 10³ kg/m³ and g = 9.81 m/sec².

Sol: The speed of transverse wave v in a string of linear density m under a tension T is given by

$v = \sqrt{T/m}$

T = 0.1 kg, wt. = 0.1 × 9.8 newton

m = cross – section area × density

$= \left(10^{-3}\right)^2 m^2 \times 9.81 \times 10^3 kg\ m^{-3}$

$= 10^{-3} \times 9.8$ kg/metre

$$\therefore \quad v = \sqrt{\frac{0.1 \times 9.8}{10^{-3} \times 9.8}} = 10 \text{ m / sec}$$

27. The fundamental frequency of vibration of a stretched string of length 1m is 256 Hz. Find the frequency of the same string of half the original length under identical conditions.

Sol: Under the constant tension and linear density n*l* = constant

$$\therefore 256 \times l = n\left(\frac{l}{2}\right) \Rightarrow n = 256 \times 2 = 512 Hz$$

28. A steel wire of 0.02 kg mass and 2m length is stretched to a tension of 2 N. What is the frequency of the fundamental vibration ?

Sol: Fundamental frequency $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$

Given, l = 2m; T = 2N; $m = \frac{0.02}{2}$ = 0.01 kg/m

$$\therefore n = \frac{1}{2 \times 2}\sqrt{\frac{2}{0.01}} = \frac{1}{4}\sqrt{\frac{200}{1}} = \frac{10}{4}\sqrt{2} = \frac{14.14}{4}$$

$\therefore n = 3.535$ Hz

29. A string of length 0.5m and linear density 0.001 kg m⁻¹ kept under a tension 1 N. Find the first three over tones of the string when it is plucked at its mid point.

Sol: The fundamental frequency n is given by $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$

Here T = 1N, l = 0.05 m; m = 0.0001 kg m^{-1} and n = ?

$$\therefore n = \frac{1}{2 \times 0.5}\sqrt{\frac{1}{0.0001}} = 100 \text{ Hz}$$

When the string is plucked at its mid point, odd overtones are present.

Hence the string vibrates with 3, 5n, 7n ….

Frequency of first overtone = 3n = 3× 100 = 300 Hz

Frequency of second overtone = 5n = 5 × 100 = 500 Hz

and Frequency of third overtone = 7n = 7×100 = 700 Hz

30. A sonometer is arranged to emit a note of frequency n. By how much the tension be varied to increase the frequency of the note to (5n/3) ?

Sol: According to second law,

$n \propto \sqrt{T}$ (l and m being the same) $\therefore n \infty \sqrt{T}$

and $\frac{5n}{3} \propto \sqrt{x}$

Eq. (2) ÷ Eq. (1), we get

$$\frac{5}{3} = \sqrt{\frac{x}{T}}$$

Squaring both sides.

$$\frac{25}{9} = \frac{x}{T} \Rightarrow x = \frac{25}{9}T$$

Hence the tension is to increase by 25/9 times the original value.

31. The fundamental frequency of a sonometer wire increases by 5 Hz., if its tension is increased by 21%. How will the frequency by affected, if its length is increased by 10%?

Sol: The fundamental frequency n is given by

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad \ldots(1)$$

When the tension is increased by 21%, the new tension will be 1.21 T and the new frequency will be (n + 5)

$$\therefore (n+5) = \frac{1}{2l}\sqrt{\frac{1.2T}{m}} \quad \ldots(2)$$

$$(2) \div (1): \frac{n+5}{n} = \sqrt{1.21} = 1.1$$

$$\Rightarrow 1.1n = n + 5 \Rightarrow 0.1n = 5 \Rightarrow n = 50 \text{ Hz}$$

When the length is increased by 10%, the new length will be (l + 0.1 l) = 1.1l and the new frequency × is given by

$$x = \frac{1}{2(1.1l)}\sqrt{\frac{T}{m}} \quad \ldots(3)$$

$$(3) \div (1): \frac{x}{n} = \frac{1}{1.1} \text{ and } n = 50Hz$$

32. Two identical strings are tuned to the same frequency of 300 Hz. The tension of one of the strings is increased by 2%. How many beats per second will be heard when the two strings are sounded together ?

Sol: Given frequency n_1 = 300 Hz

tension T_1 = T

New tension T_2 = T + (2/100) T = 1.02 T

New frequency n_2 = x

According to II law,

$n \propto \sqrt{T}$ (l and m being the same)

$$\Rightarrow \frac{n_2}{n_1} = \sqrt{\frac{T_2}{T_1}} \Rightarrow \frac{x}{100} = \sqrt{\frac{1.02T}{T}} = \sqrt{1.02}$$

$\Rightarrow x = 300\sqrt{1.02} = 303$ Hz

$\therefore$ No. of beats = 303 – 300 = 3 per sec.

33. A copper wire of radius 10^{-3} m has a length of 1 metre. It is fixed at both ends and subjected to a tension of 10^4 N. Calculate (a) the fundamental frequency and the frequency of the first two overtones (b) the corresponding wave lengths. (Density of copper = 8.92×10^1 kg. m^{-3}).

Sol: Let r be density and m mass per unit length of the wire. Then m = area of cross section × density

$$= \pi r^2 d = \pi \times (10^{-3})^2 \times (8.92 \times 10^3)$$

$$= 28.01 \times 10^{-3} kg/m$$

and T = 10^4 N

The velocity of the transverse wave is given by

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{10^4}{28.01 \times 10^{-3}}} 0.593 \times 10^3 m/\sec$$

(a) The fundamental frequency n_1 is given by

$$n_1 = \frac{V}{2l} = \frac{0.597 \times 10^3}{2 \times 1} = 298.7 Hz$$

The frequency of the first two overtones are

$$n_2 = 2n_1 = 2 \times 298.7 = 597.4 \text{ Hz}$$

and $n_3 = 3n_1 = 3 \times 297.5 = 296.1$ Hz

(b) The wave length λ_1 is given by

$$\lambda_1 = \frac{v}{n_1} = \frac{v}{(v/2l)} = 2l = 2m$$

Hence the corresponding wavelength are

$$\lambda_2 = \frac{2 \times 1}{2} = 1m \text{ and } \lambda_3 = \frac{2 \times 1}{3} = 0.667m$$

34. In Melde's experiment it was found that the string vibrated in 3 loops when 8 gm were placed in the pan. What mass must be placed in the pan to make the string vibrate in 5 loops ? (Neglect the mass of the string and pan).

Sol: In Melde's experiment

$$n = \frac{p}{2l}\sqrt{\left(\frac{T_p}{m}\right)}$$

or $T_p \times p^2 = 4l^2 mn^2 =$ constant.

If T_3 and T_5 are the tensions to produce 3 and 5 loops respectively, we have

$$T_5 \times 5^2 = T_3 \times 3^2 \text{ or } T_5 = T_3 (9/25)$$

Here T_3 = 8 gm

$$\therefore T_5 = \frac{9 \times 8}{25} = 2.88 \text{ gm wt.}$$

35. In a Melde's experiment when the tension is 100 gm, and the tuning fork vibrates at right angles to the direction of the string, the later is thrown into four segments. If now the tuning fork is set to vibrate along the string, find what additional weight will make the string vibrate in one segment.

Sol: As in the above question

$$T_p \times p^2 = \text{ constant}$$

If $T_1 \, and \, T_2$ be tension to produce one and two segments respectively, we have

$T_1 \times (1)^2 = T_p \times (2)^2$ or

$T_1 = 100 \times 4 = 400$ gm wt.

Hence additional load required= 400 – 100 = 300 gm wt.

36. In Melde's experiment in the longitudinal mode of vibration a string vibrates in 4 loops with a load of 8 cm. How much load would be required in order that the string may vibrate in 8 loops in transverse mode?

Sol: We know that

$$n = \frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$$

If L be the total length of the string and N be the number of loops, then l = L/N

$$\therefore \quad n = \frac{N}{2L}\sqrt{\left(\frac{T}{m}\right)},$$

In transverse mode,

$$n = \frac{N}{2L}\sqrt{\left(\frac{T_1}{m}\right)}.$$

In longitudinal mode,

$$2n = \frac{N_2}{2L}\sqrt{\left(\frac{T_2}{m}\right)} \qquad (\because n_2 = 2n)$$

Now $$\frac{n}{2n} = \frac{N_1}{N_2}\sqrt{\left(\frac{T_1}{T_2}\right)}$$

or $$\left(\frac{1}{2}\right)^2 = \frac{N_1^2}{N_2^2} \times \frac{T_1}{T_2}$$

Here $N_1 = 8, N_2 = 4$ and $T_3 = 8$ gm – wt,

$$\therefore \quad \left(\frac{1}{2}\right)^2 = \frac{64}{16} \times \frac{T_1}{8}$$

or $$T_1 = \frac{1}{2} gm - wt.$$

Thus load in transverse mode = 500 m gm.

37. A Piezo electric crystal has a thickness of 0.002m. If the velocity of sound waves in the crystal is 5750 ms^{-1}, calculate the fundamental frequency of crystal.

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{\vartheta}{2l} \Rightarrow \nu = \frac{5750}{2(0.002)}$

$$= 1.4375 \times 10^6 \; Hz = 1.4375 \; MHz$$

38. A Piezo electric crystal has a thickness of 3 × 10^{-3}m has density 3.5 × 10^3 kg m^{-3}. If the young's modulus of the material of the crystal 8×10^{10} Nm^{-2}, calculate the fundamental frequency of crystal.

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

$$\Rightarrow \nu = \frac{1}{2(3 \times 10^{-3})}\sqrt{\frac{8 \times 10^{10}}{3.5 \times 10^3}} = 0.7967 \times 10^6 \; Hz = 0.7967 \; MHz$$

39. A Piezo electric crystal has a thickness of 0.005m has density 2650 kg m^{-3}. If the young's modulus of the material of the crystal 7.9×10^{10} Nm^{-2}, calculate the fundamental frequency of crystal.

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

$$\Rightarrow \nu = \frac{1}{2(0.005)}\sqrt{\frac{7.9\times10^{10}}{2650}} = 0.5461\times10^{6}\ Hz = 0.5461\ MHz$$

40. *Calculate the fundamental frequency of crystal of thickness 3 mm, Y=8 × 10^10 Nm^-2, and* $\rho = 2.5\times10^3\ kg\,m^{-3}$

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

$$\Rightarrow \nu = \frac{1}{2(0.003)}\sqrt{\frac{8\times10^{10}}{2.5\times10^{3}}} = 0.943\times10^{6}\ Hz = 0.943\ MHz$$

41. *A Piezo electric crystal has a thickness of 3 mm has density 2650 kg m^-3. If the young's modulus of the material of the crystal 7.9×10^10 Nm^-2, calculate the fundamental frequency of crystal.*

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

$$\Rightarrow \nu = \frac{1}{2(0.003)}\sqrt{\frac{7.9\times10^{10}}{2650}} = 910.1\times10^{3}\ Hz = 910.1\ kHz$$

42. *Calculate the fundamental frequency of crystal of thickness 0.001m, Y= 7.9 × 10^10 Nm^-2, and* $\rho = 2650\times10^3\ kg\,m^{-3}$

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

$$\Rightarrow \nu = \frac{1}{2(0.001)}\sqrt{\frac{7.9\times10^{10}}{2650}} = 2.73\times10^{6}\ Hz = 2.73\ MHz$$

43. *The thickness of a Piezo electric crystal is 0.002m, if the velocity of the crystal is 5750 ms-1 Calculate the fundamental frequency of crystal.*

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{\vartheta}{2t}$

$$\Rightarrow \nu = \frac{5750}{2(0.002)} = 1.437\times10^{6}\ Hz = 1.437\ MHz$$

44. *Calculate thecapacitance to produce ultrasonic waves of 10^6 Hz with an inductance of 1 henry.*

So l: We have frequency $\nu = \frac{1}{2\pi\sqrt{LC}}$

$$\Rightarrow \nu^2 = \frac{1}{4\pi^2 LC} \quad \Rightarrow C = \frac{1}{4\pi^2\nu^2 L} = \frac{1}{4(3.14)^2(10^6)^2(1)} = 0.025\times10^{-12}\ farad$$

45. *A magnetostriction oscillator has a frequency 20 kHz. If it produces a sound waves of velocity 6.2 × 103 ms^-1. Find the length of ferrite rod.*

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{\vartheta}{2t}$

$$\Rightarrow l = \frac{\vartheta}{2\nu} = \frac{6.2\times10^{-3}}{2(20\times10^{3})} = 0.155\,m$$

46. Bats emit ultrasonic waves. The shortest wavelength in air emitted by a bat is about 0.33 cm. what is the highest frequency a bat can emit? velocity of sound is 330 ms^{-1}

Sol: We have frequency $\nu = \frac{\vartheta}{\lambda} = \frac{330}{\left(0.33\times10^{-2}\right)} = 10^{3}\ Hz$

❖❖❖❖❖

PRACTICAL COURSE - 1

1. Young's modulus of the material of a bar by uniform bending.

Aim: To find the Young's modulus of the given material bar by uniform bending using pin and microscope method.

Apparatus:Pin and Microscope arrangement, Scale,Vernier callipers, Screw gauge, Weight hanger, Material bar or rod.

Formula: In uniform bending, the Young's modulus of the material of the bar is given by

$$Y = \frac{mgal^2}{8Ie}$$

Where, m = Mass at each end of the bar.

a = Distance between the point of suspension of themass and nearer knife edge.

g = Acceleration due to gravity.

l = the length of the bar between the knife edges.

e = Elevation of the midpoint of the bar for a mass m at each end.

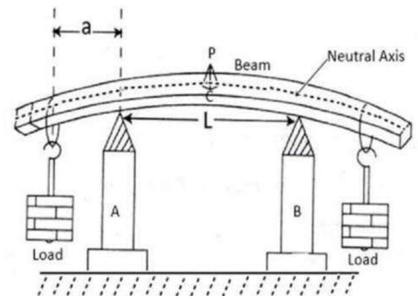
I = Geometrical moment of inertia.

For a bar of rectangular cross section, $I = \frac{bd^3}{12}$

where **b** is the breadth and **d** is the thickness of the bar.

$$\therefore \; Y = \frac{3mgal^2}{2bd^3e} \Rightarrow Y = \frac{3g\,a\,l^2}{2\,b\,d^3}\left(\frac{m}{e}\right)$$

Procedure: The beam is supported horizontally on two knife edged supports A and B symmetrically so that equal lengths of the beam project beyond the knife edges. Two loops to carry weight hangers are attached near the ends of the beam, one at each end at equal distance 'a' and 'a' from the knife edges. Two weight hangers of equal weights are suspended from these loops. A pin P is fixed vertically at the midpoint of the beam with bees wax. A travelling microscope with a vertical traverse is focused on the pin so that the inverted image of the up of the pin just coincides with the horizontal cross-wire. The reading of the microscope on the vertical scale is noted. This is taken as the initial reading for the dead load.



Two equal loads 200 gms. (or 500 gms) each, are placed on the two weight hangers. The microscope is adjusted again until the inverted image of the tip of the pin coincides with the horizontal cross wire and the reading of the microscope is noted. The experiment is repeated by increasing the loads in steps of 200 gms. each side, up to a convenient load and each time the microscope reading is taken as above. The experiment is repeated gradually decreasing the loads in same steps. Thus for each load two readings are obtained, one when increasing the load and the other when decreasing the load. The mean of the two readings is found for each load. The different between the mean reading for each load and the initial reading for the dead load gives the elevation (e) of the mid point for each load (M). Thus the elevation (e) for each load (M) is obtained.

The readings are tabulated as shown.

From the readings, $\frac{M}{e}$ value for each load 'M' is calculated, and the mean $\frac{M}{e}$ value is calculated.

The length 'l' of the beam between the knife edges is measured. The distance (a) of each weight hanger from the knife edge.

The breath 'b', and the thickness (d) of the beam are measured with a vernier calipers and a screw gauge.

The Young's Modulus is calculated from the formula.

$$Y = \frac{3gal^2}{2bd^3} \cdot \frac{M}{e}$$

A graph is drawn taking the values of 'M' on the x – axis and the corresponding values of 'e' on the y – axis. (fig.viii). It is straight line graph passing through origin. The value of $\frac{M}{e}$ is obtained from the graph.

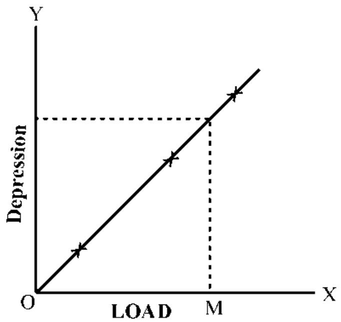


Fig.Viii

Substitute this value of $\frac{M}{e}$ in the above equation, y can be calculated.

Observations :

Length of the beam between the knife edges (l) = …. cms.

Average breadth of the beam (b) = …. cms

Average thickness of the beam (d) = …. cms

Distance between the point of suspension of the weight hanger and the knife edge (a) = …. cms.

To determine the breadth (b) of the beam using vernier calipers: LC = 0.01 cm

S.No	MSR	VC	VC × LC	**Total : MSR + (VC × LC)**
1 2 3				

Average breadth b =

To determine the thickness (d) of the beam using screw gauge:

Zero Error: ………mm. Zero Correction: ……….mm

LC = 0.01 mm

S.No	PSR	HSR	**CHSR**	**CHSR × LC**	Total : PSR+ (CHSR × LC)
1 2 3					

To determine the value of $\frac{m}{e}$:

Least count of microscope = 0.001 cm

S.No	Load M	Microscope readings			Elevation (e)	$\frac{m}{e}$
		Load increasing	Load decreasing	Mean		

Mean value of $\frac{m}{e} =$

Precautions :

1. The beam should be arranged symmetrically on the beam.
2. The microscope screw must be turned in the same direction to avoid back – lash.

Result: Young's modulus Y =

2. Young's modulus of the material of a bar by non uniform bending

Aim: To find the Young's modulus of the given material bar by non-uniform bending using pin and microscope method.

Apparatus:Pin and Microscope arrangement, Scale,Vernier callipers, Screw gauge, Weight hanger, Material bar or rod.

Formula: In uniform Bending , the Young's modulus of the material of the bar is given by

$$Y = \frac{mgl^3}{48Ie}$$

Where, m = Mass at each end of the bar.

g = Acceleration due to gravity.

l = length of the bar between the knife edges.

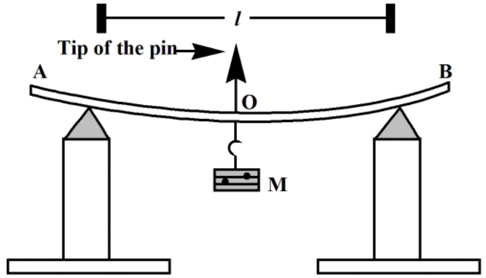
e = Depression of the midpoint of the bar for a mass m.

I = Geometrical moment of inertia.

For a bar of rectangular cross section, $I = \frac{bd^3}{12}$

Where b is the breadth and d is the thickness of the bar.

$$\Rightarrow Y = \frac{gl^3}{4bd^3}\left(\frac{m}{e}\right)$$



Procedure : The beam is supported horizontally on two knife edges A and b, symmetrically so that equal lengths of the beam project beyond the knife edges. (fig.v). The hook carrying the weight hanger is attached to the beam and is placed at the mid-point of the beam between the knife edges where a pin 'p' is fixed vertically by wax.

A travelling microscope with the vertical travers is focused on the pin so that the inverted image of the tip of the pin just coincide with the horizontal cross were and the reading of the microscope noted. The experiment is repeated by increasing the load in steps of 200 gms (or 500 gms) upto a convenient load and each time the microscope cading is taken as above. The experiment is repeated gradually decreasing be load in the same steps. Thus for each load two readings are obtained. One is for increasing the load and the other for decreasing the load. The mean of these two readings is found for each load. The difference between the mean reading for each load and the initial reading or the dead load gives the depression 'c' for each load M. Thus the depression '*e*' for each load 'M' is obtained. The readings are tabulated as shown.

From the readings, $\frac{M}{e}$ value for each load 'M' is calculated. Mean value of $\frac{M}{e}$ is calculated.

The length '*l*' of the beam between the knife edges is measured. The breadth of the bam 'b' and the thickness of the beam'd' are measured with in Vernier calipers and a Screwguage.

The Young' Modulus is calculated from the formula

$$y = \frac{gl^3}{4bd^3} . \frac{M}{e}$$

Graph: The graph is drawn taking the values of 'M' on x-axis and the corresponding values of 'e' on the y – axis fig. It is a straight line graph through origin. The value of $\frac{M}{e}$ is obtained from the graph substituting this value of $\frac{M}{e}$ in the above equation can be calculated.

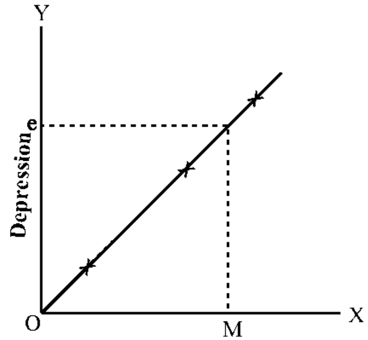


Fig.vi

Precautions :

1. The beam should be arranged symmetrically on the knife edges.
2. The microscope screw must be turned in the same direction to avoid back – lash.

To determine the breadth (b) of the beam using vernier calipers: LC = 0.01 cm

S.No	MSR	VC	VC × LC	TOTAL : MSR + (VC × LC)
1 2 3				

Average breadth b =

To determine the thickness (d) of the beam using screw gauge:

Zero Error : ……….

Zero Correction : ……….mm

LC = 0.01 mm

S.No	PSR	HSR	CHSR	CHSR × LC	Total : PSR+ (CHSR × LC)
1 2 3					

To determine the value of $\frac{m}{e}$:

Least count of microscope= 0.001 cm

S. No	Load M	Microscope readings			Elevation (e)	$\frac{m}{e}$
		Load increasing	Load decreasing	Mean		

Mean value of $\frac{m}{e}$ =

3. Surface Tension of a Liquid by Capillary Rise Method

Aim: To determine the surface Tension of water by measuring its rise in capillary tube.

Apparatus: A capillary tube of uniform bore, travelling microscope, a beaker, a fine wire bent twice at right angles and a stand with clamp.

Principle: When a capillary tube open, at both ends is dipped vertically in a liquid like water, which wet glass, the liquid level rises into the tube due to surface tension. In the case of

liquids like mercury, which do not wet glass, the liquid level is depressed.

If 'h' is the height of the liquid level in the tube above the outside liquid level and r is the radius of the capillary tube the surface tension T is given by.

$$T = \frac{1}{2}\left(h + \frac{1}{3}r\right) \text{ r } \rho \text{ g dynes/cm.}$$

r = radius of capillary bore ρ = density of water

h = capillary rise

Procedure: A capillary tube of narrow uniform bore is taken and is thoroughly cleaned with acidified potassium dichromate solution to remove traces of grease or oil. It is then washed with tap water and dried. A piece of wire is bent twice at right angles and attracted to the tube with rubber bands. The inner side of the tube is wetted with water and the tube is then clamped vertically so that its lower end dips few centimeters inside the water contained in a beaker placed on a wooden block or on an adjustable stand. (fig. ix). The bent wire is adjusted so that its tip just touches the surface of water in the beaker.

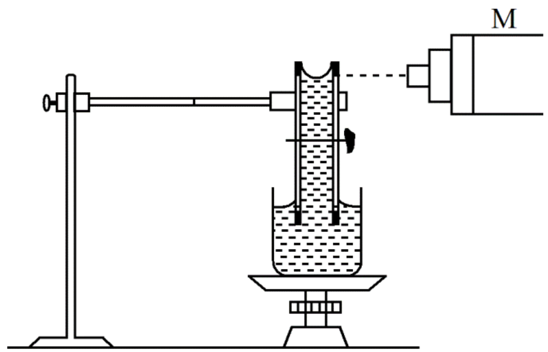


Fig.ix

a) To determine 'h': A travelling microscope capable of vertical motion is then focused on the lower meniscus of water in the capillary tube so that the horizontal cross wire coincides with the lower surface of the meniscus which appears inverted. The microscope reading is noted. The beaker containing water is removed from the below the tube without disturbing the tube. The microscope is lowered and focused on the tip of the wire so that the tip coincides with the horizontal cross wire. The reading of the microscope is noted. The difference between the two reading gives the height 'h' of the water in the capillary tube above the free surface of the liquid outside it.

The experiment is repeated two or three times by immersing the tube to different depths in water. The Mean value is noted.

b) To determine radius of the capillary bore: The tube is removed and i.e., clamped horizontally. The microscope is focused on the capillary bore. The vertical cross wire is made to coincide with one end of the bore and the reading is noted. The microscope is moved horizontally and the vertical cross wire is again made to coincide with the other end of the bore. The reading is noted, the difference between the two readings gives the horizontal diameter of the bore. Similarly the vertical diameter is also determined by making the horizontal cross wire coincide with the two ends of the bore and moving the microscope vertically. The mean of the two diameters is calculated from which the average radius '*r*' of the capillary bore is calculated.

The temperature of the water in the beaker is noted.

The surface tension is calculated from the formula.

$$T = \frac{1}{2}\left(h + \frac{r}{3}\right) r \rho g$$

Observations:

Least count of microscope = 0.001 cm

S.No.	Reading of the microscope		h = a - b
	Lower surface of the meniscus (a)	Pointed and of the wire (b)	

Mean h = ………………

Temperature of water = ………………….ºC

Mean radius of the capillary tube = …………… cms

Density of water at room temperature = is ………..gm/cc.

Precautions :

1. The capillary tube should have narrow uniform bore.
2. It should be clean and dry.
3. The tube must be clamped vertically.

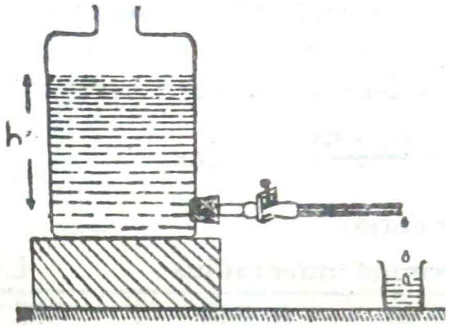
Result: Surface tension of the given liquid = dynes/cm.

4. Viscosity of Liquid by the Flow Method (Poiseuille's Method)

Aim: To determine the coefficient of viscosity of water by Poiseuille's method.

Apparatus: A constant pressure head, a uniform capillary tube, beaker, watch-glass and stop-watch.

Description: The arrangement of the apparatus consists of an aspirator bottle of about two litres capacity provided with an opening at the side near the bottom. The opening is closed by one - hold rubber stopper through which passes a short glass tube. The capillary tube is connected to the outer end of the glass tube with a short rubber rubbing which is provided with a pinch-cock. When the aspirator is filled with water, it flows out through the capillary tube which is kept in a horizontal position When the collection of water is over, the flow of water can be stopped by closing the pinch-cock.



If h_1 and h_2 represent the heights of water level in the aspirator bottle above the axis of the capillary tube before and after the collection of water, the mean pressure exerted is given by P = hdg where h is the average height [= $(h_1 + h_2)/2$], d the density of water and g the acceleration due to gravity.

Procedure: The capillary tube is cleaned well first with acidified potassium dichromate solution and then with tap water. It is then fixed to be upper tube and clamped with its axis horizontal. A short length of fine thread is tied to tube free end of the capillary tube so that the water coming out of the capillary tube trickles down along the thread in drops.

A clean and dry beaker is taken. It is then placed under the free end of the capillary tube. The height h_1 cm. of water level in the aspirator bottle above the axis of the capillary tube is measured with a metre scale. By opening pinch-cock

completely water is allowed to flow through the capillary tube into the beaker for a sufficient interval of time t seconds (about 15 minutes). The pinch-cock is closed and the height h_2 of the water level in the aspirator bottle above the axis of the tube is measured.

Now the water is poured in a measuring jar and its volume is noted from it. By using the travelling microscope the internal radius of the capillary tube ‘r’ is noted.

Formula :

Coefficient of Viscosity $\eta = \frac{\pi P r^4 t}{8 V l}$

dyne/cm^2/unit velocity gradient or Poise

l = length of capillary tube = cm.
r = internal radius of capillary tube = cm
V = volume of water collected in ‘t’ sec. = *ml*.
P = hdg = dyne / cm^2.
h_1 = initial height of water level = cm
h_2 = final height of water level = cm.
d = density of water = 1 gm/ cc

$h = \frac{h_1 + h_2}{2} =$ cm.

Tabular Forms :

To determine inner radius : **L.C. = cm.**

	M.S.R. (a) cm.	V.C.	V.C×L.C. (b) cm.	Total (a+b) cm
R_1				
R_2				

Inner radius = $\left(\frac{R_1 \sim R_2}{2}\right) cm$.

S.No.	h_1	h_2	$h = \frac{h_1 + h_2}{2}$	Vol. of water (ml.)	Time (t) sec.

Precautions :

1. The flow of water must be steady.
2. The pinch clamp should be opened completely So that the water flows under atmospheric pressure only.

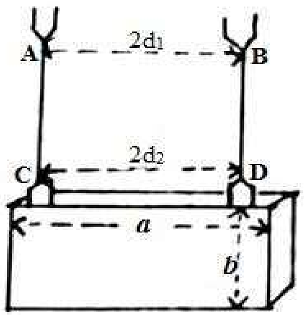
Result: Coefficient of viscosity $\eta =$ Poise.

5. Bifilar suspension – moment of inertia of a regular rectangular body

Aim: To determine the moment of inertia of a regular rigid body by means of Bifilar suspension.

Apparatus: A regular body like a rectangular block, two wires, meter scale, stop-watch, rough balance.

Description: The arrangement consists of a regular rigid body like a rectangular parallelepiped. The body contains two screws C and D at its top for suspension. It can be suspend from a stand provided with two hooks A and B in the same horizontal line. The distance between the upper and the lower screws are fixed. But in some arrangements one of these distances can be changed.



Procedure: The mass of the regular body 'm' is determined with rough balance. The distance between A and B ($2d_1$) and C and D ($2d_2$) are noted. (The length of the wires AC and BD are noted). The body is made to vibrate in the horizontal plane. The time taken by the body for 20 oscillations is noted using a stop-clock. The time for one oscillation or the time period (T) is determined. The experiment is repeated by changing the length of the wire (l). The readings are tabulered. The practical value of Moment of inertia is compared with theoretical value.

Formula:

Moment of Inertia:

Practically : $I = \frac{mg}{4\pi^2}\frac{d_1 d_2 T^2}{l} gm-cm^2$

Theoritically: $I = m\left(\frac{a^2+b^2}{12}\right)$ (for rectangular block only)

m = mass of the body = gm

$2d_1$= distance between upper screws = cm.

$2d_2$= distance between lower screws = cm.

T = time period = sec.

l = length of the wires = cm.

a = length of the block = cm.

b = breadth of the block = cm.

Tabular form:

S.No.	d_1	d_2	Length of The wire (l)	Time for 20 oscillations			Time for One oscillation (T)	$\frac{d_1 d_2 T^2}{l}$
				Trial I	Trial II	Mean		

Mean value $\frac{d_1 d_2 T^2}{l} =$

Precautions :

1. The body should make horizontal oscillations.
2. The body should not oscillate in the vertical plane.

Result : Moment of Inertia :

Practically I = gm – cm^2

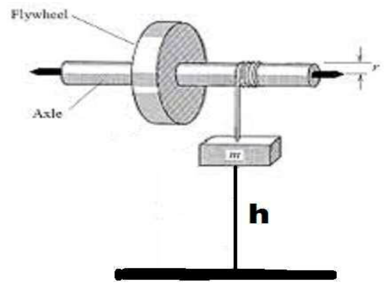
Theoretically I = gm – cm^2

6. Fly - Wheel – Determination of Moment of Inertia

Aim: To determine the moment of inertia of a flywheel about its axis of rotation.

Apparatus: Flywheel, Meter scale, Vernier Calipers, Stop watch, known mass with hooks.

Description: The fly-wheel consists of a massive wheel W with a long horizontal axle on either side supported on ball bearings embedded in a bracket. The base of the bracket is fixed to a rigid support like a wall or a work table.



A short peg projects from the axle. The free end of a string carrying a mass m is looped over the peg and the length of the string is so adjusted that the loop slips off the peg automatically when the mass touches the ground. The

flywheel is turned so that the string is wrapped round the axle evenly without any overlapping.

Procedure: When the mass just touches the ground a chalk mark is made on the rim of the wheel and another chalk mark is made exactly opposite to it on the bracket. These two marks help us to count the number of revolutions made by the wheel.

The flywheel is turned so that the string is wrapped round the axle evenly n_1 times without any overlapping. The height h between the ground and the bottom of the mass is measured with a scale. The wheel is released and allowed to rotate under the action of the mass m. A stop watch is started just when the loop slips off the peg. The time t and the number of revolutions n_2 made by when before it comes to rest are noted. The experiment is repeated twice and the mean values of n_2 and t are found. The angular velocity is calculated ω using the relation $\omega = \frac{4\pi n_2}{t}$.

The diameter of the axle is measured in two mutually perpendicular directions by means of a vernier calipers. The mean diameter and hence the mean radius r of the axle is found.

The moment of inertia of the flywheel about its axis of rotation is calculated from the equation.

$$I = \frac{n_2 m}{n_1 + n_2}\left[\frac{2gh}{\omega^2} - r^2\right]$$

The experiment is repeated for two or three different values of h and the average moment of inertia of the flywheel is determined.

Formula:

$$I = \frac{n_2 m}{n_1 + n_2}\left[\frac{2gh}{\omega^2} - r^2\right]$$

Where m = mass attached to the string

n_1= no. of revolutions before detachment of the mass

n_2= no. of revolutions after detachment of the mass.

h = height of the mass above the ground.

ω = Angular velocity = $\frac{4\pi n_2}{t}$ rad/sec.

r = radius of the axle g = 980 cm/sec^2.

Tabular forms:

To determine the radius of the axle:

$$\text{L.C.} = \frac{\text{Value of 1 main scale division}}{\text{Total no. of vernier scale division}} = \frac{S}{N} = \frac{0.1}{10} = 0.01 \text{ cm}$$

S.No.	M.S.R. (a) cm.	V.C.	V.C × L.C. (b) cm.	Total (a+b) cm.

Average diameter of the axle = cm

Average radius of the axle = cm

To determine the value of I:

S.No.	Mass (m) gm	h	No. of revolutions n_2			Time for n_2			$\omega = \frac{4\pi n_2}{t}$	I $gm-cm^2$
			Trial I	Trial II	Mean	Trial I	Trial II	Mean		

Average value of I = gm-cm^2

Precautions:

1) The string should be wound round the axle evenly without any overlapping.
2) The loop slipped on the peg should be loose.
3) The flywheel should start of its own accord under the action of the mass attached to it without giving any push.

Result: The moment of inertia of the fly wheel, I = gm-cm^2

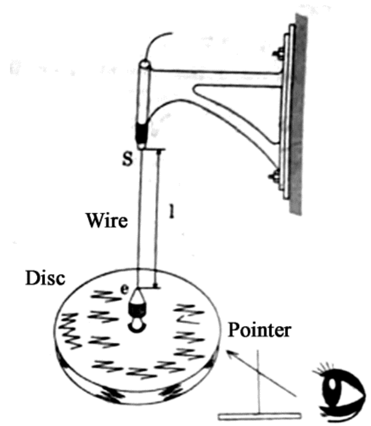
7. Rigidity modulus of material of a Wire-Dynamic Method (Torsional Pendulum)

Aim: To determine the rigidity modulus (η) of the material of a given wire using torsional pendulum.

Apparatus: The wire in the arrangement of a torsional pendulum, pointer, sensitive stop clock, vernier calipers, screw gauge, metre scale.

Formula: Rigidity modulus (η) of the material of the wire made as a torsional pendulum is given by,

$$\eta = \frac{4\pi MR^2}{a^4}\left(\frac{l}{T^2}\right)\frac{dynes}{cm^2}$$



Here

M = Mass of the disc (gm)

R = Radius of the disc (cm)

l = length of the wire (cm)

a = radius of the wire (cm)

T = time period of oscillation

Description: A torsional pendulum is as shown in fig. It consists of a disc hung by means of a long wire of uniform cross section. The thickness of the disc will be very small compared to its diameter. The disc is suspended by the wire passing through its centre C and is tightly fixed by chuck nut. The upper end of the wire is suspended from a rigid support by passing the wire through another chuck nut as shown in the figure. The disc hangs in a horizontal position. The disc is usually made of a metal like brass. A vertical line mark is made on the side (thickness) of the disc and the pointer is placed vertically in front of this mark. We place our eye behind the pointer and count the oscillations of the disc by noting the crossings of the line mark. Keeping the wire in its position, if we draw the disc rotated through a small angle and release it, then due to the twist developed in the wire torsion) the disc will be making oscillations in a horizontal plane about the wire as its axis of rotation These oscillations are torsional oscillations and hence the arrangement is called a torsional pendulum.

We leave out the first few oscillations. Then, as the line mark crosses the pointer from left to right we count it as zero (oscillation) and start the stop clock. When the line mark once again crosses the pointer once again in the same direction (from left to right) we count it as one (oscillation). In this manner we can count about 20 oscillations and find the time period (T).

Experimental Procedure: The disc issuspended by means of the given wire to form a torsional pendulum as shown in Fig. With the help of the check nuts, we first keep the length of the wire*l*= 50 cm. A vertical line mark is drawn on the side

(thickness) of the wire and we keep the pointer vertically in front of the line mark. Then we rotate the disc slightly so that the wire gets twisted (by <5). Now we release the disc and allow it to make simple harmonic oscillations.

Observing the line mark from behind the pointer and with the help of the stop clock we note down the time taken for 20 oscillations.

Without changing the length, the experiment is once again repeated and the time taken for 20 oscillations is noted for a second time. The average value of these two is divided by 20 to get the time period of oscillation T.

Next, the length of the wire is increased by 10 cm, that is l = 60 cm and the same procedure as above is repeated. In this way, the experiment is repeated for 4 or 5 different lengths. The readings are entered in the tabular form.

Each time, the length l can be measured with a meter scale. The diameter of the disc is measured with a Vernier calipers and the diameter of the wire is measured with a screw gauge. The mass M of the disc is found with a rough balance.

From the table, average value of $\frac{I}{T^2}$ is calculated. A graph is drawn between $l$and T^2 and $\frac{l}{T^2}$ value is found from the graph also.

These values are substituted in formula to get the rigidity modulus 'η' of the material of the given wire.

Observations: (1) l – T values and $\frac{l}{T^2}$ values

S.No.	Length of the Wire l cm	Time taken for 20 oscillations t (s)			Time period $T = \frac{t}{20} s$	$\frac{l}{T^2}\left(\frac{cm}{s^2}\right)$
		Trial I	Trial II	Mean t		
(1)	50 cm					
(2)	60 cm					
(3)	70 cm					
(4)	80 cm					

Average value of $\left(\frac{l}{T^2}\right)$ = cm/s^2

To determine the breadth (b) of the beam using Vernier calipers: LC = 0.01 cm

S.No	MSR	VC	VC × LC	Total : MSR + (VC × LC)
1 2 3				

Average breadth = cm

Average radius R = cm

To determine the radius (a) of the wire using screw gauge:

Zero Error (ZE): …….. Zero Correction (ZC): ……mm

LC = 0.01 mm

S.No	PSR (a) mm	HSR	CHSR	CHSR × LC b mm	Total : (a+b) mm
1 2 3					

Average diameter = mm

Average radius a = mm

Result: Rigidity modules of the material of the wire

$$\eta = \frac{4\pi MR^2}{a^4}\left(\frac{l}{T^2}\right) dynes/cm^2$$

$$\eta = \left(\frac{n}{10}\right) = \quad newton/m^2$$

Precautions:

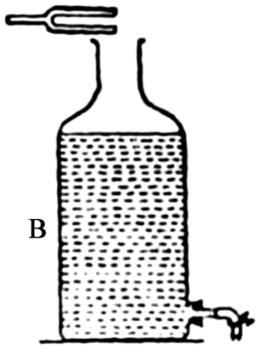
1) There should be no kinks any where is the wire.
2) The diameter of the wire is to be carefully measured with a screw gauge at least at six different places and the average is to be calculated.
3) The diameter of the disc should also be carefully measured with Vernier calipers at least at four different places and the average is to be calculated.

8. Volume Resonator Experiment

Aim: To verify the relation between the volume of air in the resonator and the frequency of the note that produces resonance with it.

Apparatus: Aspirator bottle, tuning forks of different known frequencies, a trough and a measuring jar.

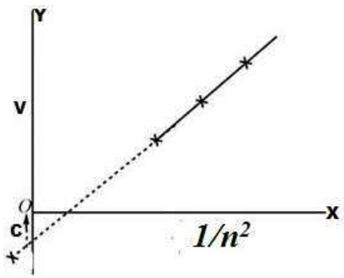
Description: The usual type of Helmholtz resonator used for laboratory work consists of an aspirator bottle B of about 2 litre capacity with a narrow cylindrical neck at the top and an opening at its side near the bottom. The opening is closed with a one-holed rubber stopper through which passes a glass tube. A short rubber tube provided with a pinch-cock is connected to the free end of the glass tube.



Procedure: The pinch - cock is closed and the aspirator bottle is filled with water up to its neck. A tuning fork of frequency n is excited and held over the mouth of the bottle with the plane of vibration of the prongs vertical. The pinch-cock is opened and water is allowed to flow out of the bottle slowly in a continuous stream into a beaker till the maximum sound is heard. The pinch-cock is then closed. In this position the natural frequency of air inside the bottle is equal to the frequency n of the tuning fork. The volume of air inside the bottle is found by adding water to the aspirator bottle up to the bottom of the neck from a measuring jar. The experiment is repeated two or three times with the same tuning fork and the mean volume v of air inside the bottle resonating with the tuning fork of frequency n is found.

The experiment is repeated with two or three tuning forks of different known frequencies. In each are the volume of air resonating with the tuning fork is found.

A graph is drawn with v along the Y-axis and $1/n^2$ along the X-axis. The graph will be a straight line cutting the ordinate beyond the origin. The intercept OA gives the neck correction C.



Formula: The relation between volume of air in the resonator and

the frequency of the fork '*n*' is $n^2 (v + c)$ = constant

Unknown frequency $n = \sqrt{\dfrac{A}{(v+c)}}$

A = average value of $n^2(v+c)$

Tabular form:

S.No.	Frequency of the fork (n) Hz	Volume of resonating air (V)			n^2	$1/n^2$	$n^2(v+c)$
		Trial I	Trial II	Mean(v)			

Precautions:

1) The tuning fork should be held only by its shank.
2) The excited tuning fork should be kept over the open end of the tube with the plane of vibration of the prongs vertical.

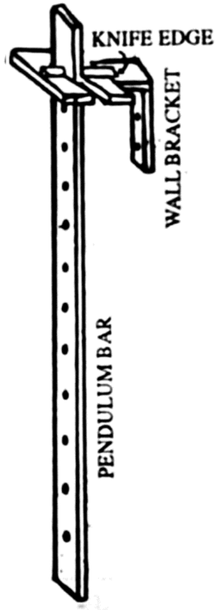
Result: The value of n^2 (v+c) is found to be constant within the limits of experimental errors.

9. Determination of 'g' by Compound Bar Pendulum

Aim: To determine the acceleration due to gravity 'g' using a compound pendulum.

Apparatus: The compound pendulum, stop-watch, knife-edge, metre scale and a telescope.

Description: The compound pendulum consists of a uniform rectangular brass or iron bar with a number of holes drilled at regular intervals along the length of the bar. The pendulum can be suspended vertically by means of a horizontal knife edge passing through any one of the holes. The knife-edge rests on a plane horizontal rigid support.



Procedure :Two fine chalk lines are marked one at each end and parallel to the length of the pendulum. The pendulum is suspended at the hole nearest to one end and the distance between the knife-edge and that end is measured. The telescope is placed on a stool at a distance of about one metre from the pendulum and pointed towardsthe lower end of the pendulum. It is then focussed on the chalk line so that the point of intersection of the cross-wires of the telescope coincides with the chalk line. This coincidence serves as a reference point to count the oscillations of the pendulum.

The pendulum is displaced slightly to one side and released so that it oscillates with a small amplitude in the vertical plane without any wobbling i.e., without any twist as it oscillates. The time taken for 50 oscillations is found twice and the mean period T is determined. The pendulum is then suspended from each hole in turn and in each case the period and the distance of the point of suspension from the same end are measured.

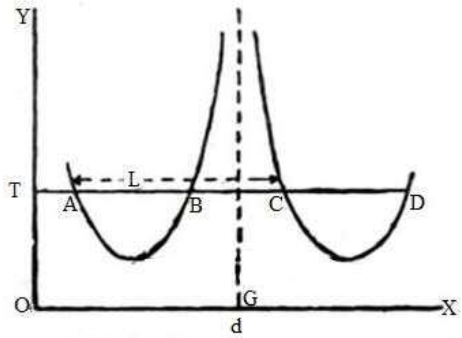
On approaching the C.G. of the pendulum the period becomes very large.

The pendulum is then reversed and the experiment is repeated by suspending the pendulum from each hole till the other end is reached. It should be noted that, even after

reversal, the distance of the knife-edge from the same end should be measured.

The pendulum is balanced on a knife-edge and the position of the C.G. of the pendulum is located. The distance of the C.G. is then measured from the same end.

A graph is drawn with period along the Y-axis and distance from one end along the X-axis. The graph is of the form shown in the fig. If OG represents the distance of center of gravity from one end, then the line drawn through G parallel to the Y-axis divides the graph into two symmetrical portions.



A line is drawn parallel to X-axis cutting the curve at four points. A,B,C and D which have the same period T. The points A, C and B, D are two sets of interchangeable points having the same period T. Hence the length AC or BD gives the length L of the equivalent simple pendulum. Actually the mean of AC and BD is taken as L.

Three or four lines similar to ABCD are drawn parallel to the X-axis. In each case the period and the corresponding length of the equivalent simple pendulum are noted.

Formula :

$$g = 4\pi^2 \frac{L}{T^2}$$

Where g = Acceleration due to gravity
L = Equivalent length of the pendulum
T = time taken for one oscillation

Tabular forms:

S.No.	Distance of knife edge from one end (d)	Time for 20 oscillations			Period T = x/20
		Trial I	Trial II	Mean (x)	

To determine (L/T^2) from graph:

S.No.	Time Period T	T^2	Equivalent length			L/T^2 Cm/sec^2
			AC	BD	L = (AC+BD)/2	

Average of $\frac{L}{T^2}$ =

$g = 4\pi^2 \frac{L}{T^2} =$ cm/sec^2

Precautions:

1) The knife edge should be perfectly horizontal.
2) The pendulum should oscillate in a vertical plane with a small amplitude and without any wobbling.

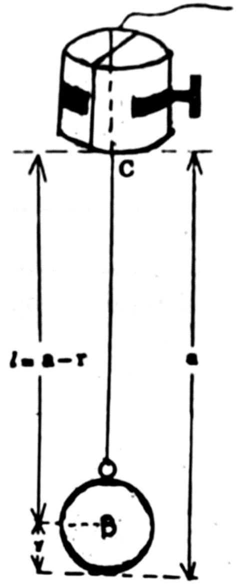
Result: Acceleration due to gravity (g) = cm/sec^2

10. Simple Pendulum – Normal distribution of Errors – Estimation of time period and the error of the mean by statistical analysis

Aim: To determine the standard error and most probable error in determining acceleration due to gravity.

Apparatus: Thread, bob, meter scale, stop – watch, calipers.

Description :The simple pendulum consists of a small metallic bob suspended by a torsion less inextensible string. The upper end of the string is passed through a split-cork fixed tightly in the clamp of a retort stand. The distance between point of suspension and the centre of the bob is called length of the pendulum.



Procedure:By using Vernier Calipers the radius of the bob*(r)* is determined. The distance between the point of the suspension and the bottom end of the bob (*a*) is measured. Then the length of the pendulum *l= (a-r)*. Now the bob is made to oscillate with smaller amplitude. Then it makes simple harmonic oscillations. The time for 20 oscillations is noted using a stop watch. The experiment is repeated fordifferent values of pendulum length (*l*). The readings are tabulated in the tabular form.

Formula:

Error 'e' = $\left(\text{observed value of } \frac{l}{T^2}\right) - \left(\text{man value of } \frac{l}{T^2}\right)$

Standard error = $\sqrt{\frac{\sum e^2}{\sqrt{n(n-1)}}}$

n = number of observations considered

Probable error = 0.6745 × standard error.

Standard value of $\left(\frac{l}{T^2}\right) =$

Mean value of $\frac{l}{T^2} \pm$ Probable error

$g = 4\pi^2 \frac{l}{T^2} cm/\sec^2$.

Tabular forms:

S.No.	Length of the Pendulum *l* cm	Time for 20 oscillations			Time period T Sec.	T^2	$\frac{l}{T^2}$
		Trial I	Trial II	Mean			

Mean value of $\frac{l}{T^2} =$

S.No.	$\frac{l}{T^2}$	error (e)	e^2

$n = \quad ; \sum x^2 =$

(All the observations may be corrected to the first decimal place.)

Precautions:

1) The oscillation amplitude should be small.
2) The length of the pendulum is noted properly.

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